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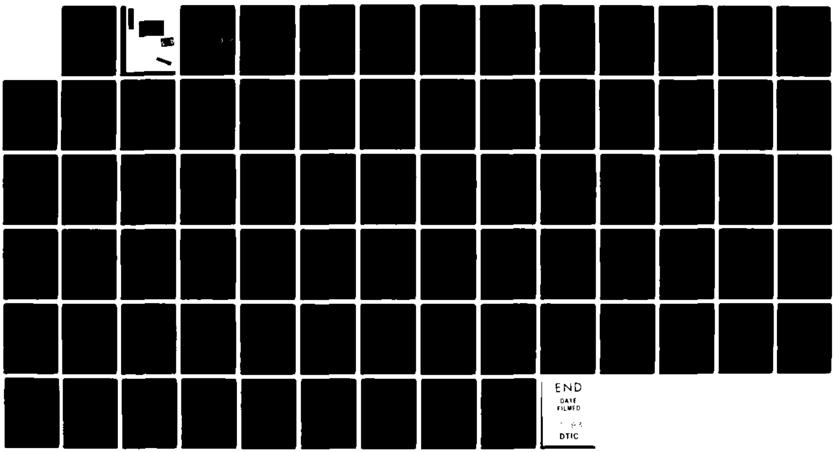

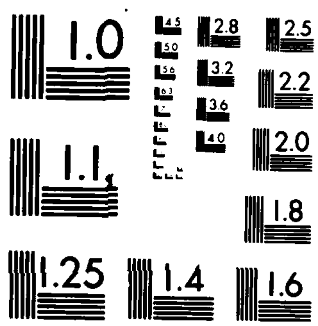

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THE AEMS PROJECT IN APPLIED 12

MATHEMATICS AT

CARNEGIE-MELLON UNIVERSITY

Comprehensive Final Report

July 1951 - December 1982

THE AEMS PROJECT IN APPLIED

MATHEMATICS AT

CARNEGIE-MELLON UNIVERSITY

12

Comprehensive Final Report

July 1951 - December 1982

prepared for

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U.S. Army Research Office
Research Triangle Park, North Carolina

by

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This is a history of the research project:

Analysis of Electrical and Mechanical Systems

for the thirty year period, July 1951 - December 1982.

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THE AEMS PROJECT IN APPLIED MATHEMATICS
AT CARNEGIE-MELLON UNIVERSITY

This is a record of a research project in applied mathematics at Carnegie-Mellon University. The project is called "Analysis of Electrical and Mechanical Systems" or AEMS for short. Since 1951 this project has received continual support from the Army Research Office, Research Triangle Park, North Carolina.

The goals of AEMS were threefold:

(a) to apply mathematical analysis to treat new problems arising in science and technology;

(b) to develop new mathematical concepts and structures using facets of science and technology as models; and

(c) to help students start a career in the interesting area of applied mathematics.

The records listed here give evidence that, in some measure, these goals have been met.

ANALYSIS OF ELECTRICAL AND MECHANICAL SYSTEMS

Report Contents

I. Personnel Involved in the AEMS Project

The Director of the AEMS project is R. J. Duffin, University Professor of Mathematical Sciences at CMU. Dr. Duffin received the B.S. degree in engineering and the Ph.D. degree in physics both from the University of Illinois. He serves as associate editor for several journals of applied mathematics. He is a member of the National Academy of Science and of the American Academy of Arts and Sciences. He is a part-time consultant to the Westinghouse Research Laboratories. His teaching duties at CMU involve undergraduate lecture courses in applied mathematics and supervision of Ph.D. students. In 1982 Dr. Duffin was awarded the John von Neumann Theory Prize by the Operations Research Society and the Institute of Management Science.

Over the years, the AEMS project has involved a large number of people in various support roles. These people may be classified as follows:

(1) Administrators both at AROD and at CMU have devoted a lot of time to initiating, evaluating, and running this project. Their support has been appreciated.

(2) Many clerical workers have been involved in preparation and handling of reports. In this connection, mention must be made of the CMU Mathematics Office and the Information Processing Office at AROD.

(3) Various faculty members of CMU have served as senior investigators for AEMS. Such faculty were selected on the basis of being productive researchers in areas overlapping the research

areas of AEMS. It was possible to provide some of these investigators with salary for two summer months.

(4) Many graduate students served as research assistants on this project. Those selected were excused from tuition and were given financial support. Such graduate students have produced a significant part of the scientific contribution of the AEMS research. The students who have done their research in this area and who have received the doctorate are listed in Section II.

(5) Much of the research of AEMS has been in collaboration with people not supported by the contracts. In fact, many were not at CMU. A list of all scientists who have been collaborators is given in Section III.

II. Doctorates Awarded

The AEMS project helped the following students to complete their Ph.D. theses in applied mathematics:

Raoul Bott	Les Karlovitz
Hans Weinberger	Moshe Mangad
Thom Greene	Merrill Patrick
George Baldwin	Elmor Peterson
Russell Meussner	Thomas Porsching
Douglas Shaffer	Norman Morrison
William Warner	Alan Washburn
William Berger	Mary Winter
Elsa Keitzer Boyce	Joan Rohrer Hundhausen
John Dettman	David McLain
Thomas Elkins	William N. Anderson
Edwin Farr	John Lackmann
Robert Gast	Donald Taranto
Carl Zorowski	Richard McDermott
Robert Helmbold	George Trapp
Eugene Shelly	Maurice Weir
Kenneth Kretschmer	S. Bhargava
Gary Kurowski	Victor Burke
William Serbyn	Patrick Hayes
Charles Duris	Thomas Morley
Edwin Rogers	Jeff Buckwalter
Harry Nain	George Polak
Dietmar Borchers	

III. Scientists Participating in the Research

Scientists who have collaborated on AEMS projects:

E. W. Adams
R. Alo
W. N. Anderson
P. Andrews
M. F. Barnsley
D. C. Benson
S. Bhargava
C. E. Blair
R. Boyer
P. J. Brehm
J. Buckwalter
V. Burke
C. V. Coffman
B. D. Coleman
A. J. Das
M. DeGroot
R. J. Duffin
A. Federowicz
H. Feshbach
J. M. Greenberg
H. J. Greenberg

P. R. Gribik
P. Gustafson
D. Hazony
A. E. Heins
T. R. Jefferson
R. G. Jeroslow
L. A. Karlovitz
G. P. Knowles
I. Kolodner
K. O. Kortanek
M. Lavie
D. N. Lee
A. D. Martin
R. A. MacCamy
J. McWhirter
V. J. Mizel
T. D. Morley
Z. Nehari
T. Nishizeki
W. Noll
J. J. Oravec

E. L. Peterson
G. Polak
M. J. Rao
F. N. Rhines
E. Saibel
A. Schild
T. W. Schmidt
J. S. Schruben
D. H. Shaffer
T. Shimpuku
S. Silver
R. Smith
W. Stokey
B. Swanson
G. E. Trapp
S. Tu
P. Whidden
S. B. Wilson
O. Wyler
Z. Zener

IV. History of the AEMS Project (1950-1960)

Prior to the second world war Professor Richard Duffin and Professor Albert Heins both taught mathematics at Purdue University. During the war Duffin was at the Carnegie Institute of Washington headed by Dr. Vannevar Bush. Heins was at the M.I.T. Radiation Laboratory. Out of this experience they both came to the belief that the Federal Government should continue to support science in peace time.

In particular, Duffin and Heins felt that applied mathematics was in special need of support because American universities had placed very little emphasis on this area of science. With this thought in mind and with the help of Professor Herbert Greenberg they drafted a proposal for a research project. The goal of this project was not only to solve certain research problems but also to interest graduate students in the area of applied mathematics as a career.

In 1950, Carnegie Institute of Technology submitted a proposal to the Office of Ordinance Research at Durham requesting support for various projects in mathematics and mechanics. The upshot was that the OOR awarded a contract for the AEMS project and named Duffin and Heins as principal investigators. The research work began 1 July 1951 under contract number DA-36-061-ORD-113.

The original proposal of Duffin and Heins suggested research in the area of mathematical analysis relating to electromagnetic waves, mechanical vibrations, and electrical

networks. The mathematical techniques to be applied were Fourier analysis, complex variable methods, and functional analysis. Over the years, the AEMS research has continued to follow these main lines. Nevertheless the research paths have widened and branched in order to relate effectively with new areas of science and engineering.

The contract provided for associate and assistant investigators besides principal investigators. Qualified graduate students were selected as research assistants. The contract had the following immediate effects:

(1) Teaching loads for the principal investigators were reduced by one three-hour course during the school year.

(2) Two months' salary was provided for full-time research during the summer.

(3) Clerical assistance was provided for preparing research reports.

(4) Students selected as research assistants were thereby provided with the financial support needed to go on for graduate degrees.

(5) The University was granted overhead expenses.

(6) The research output rate more than doubled. (Presumably this is because the goals and deadlines of a contract add further incentive to a researcher's natural curiosity.)

Under contract 113, the AEMS project prospered: research was carried out, technical reports were prepared, and research assistants received degrees in applied mathematics. As a result, the project was supported by new contracts,

DA-36-061-ORD-277, 378, and 490. Contract 490 terminated 31 August 1961. During this period, over fifty technical reports were prepared under the contracts. These are listed in Section VI.

When the AEMS proposal was submitted to OOR, it was also submitted to the Wright Air Development Center (WADC). It resulted that the Wright Field Analytical Section wished to support part of the AEMS project having to do with mechanical vibrations. Because of this the AEMS project was split into two areas. The OOR contract supported the major area, and the WADC contract supported the minor area (together with another research project in mechanics). This Wright Field work was quite successful. However, the Defense Department later ruled that Wright Field could not sponsor such university research projects. This function was given to the Office of Scientific Research. Thus this support came to an end. The technical reports relating to AEMS and prepared under the WADC contracts AF-33(616)-1 and AF-33(616)-294 are listed in Section VII.

In the academic year 1955-56, Heins was on sabbatical leave in Denmark. This enabled him to attend several scientific meetings in Europe. When he returned, his research took on new directions because of this experience.

In September 1958, Duffin took a leave of absence from Carnegie Institute of Technology to serve as Director of Special Research in Applied Mathematics at Duke University. This was sponsored by OOR under Contract DA-31-124-ORD. This

position permitted him to visit Government research laboratories throughout the United States to seek out current problems in applied mathematics. Of the various problems found, three proved to be of major interest. The first of these concerned the stress analysis for a large rocket. Another involved mutual capacitance of a system of three conductors; this was solved by the classical potential theory. A third problem concerned a new prediction formula for time series and led to a series of papers. The main technical reports prepared at Duke are listed in Appendix F.

In 1959-60 Duffin served as visiting professor at the Dublin Institute for Advanced Studies. While there, he developed a new formulation of particle dynamics which is termed "pseudo-Hamiltonian mechanics". The experiences at Dublin gave the AEMS new ways to branch out.

V. Some Research Themes of the Project

Described here are a selection of research themes of the AEMS project. As we mentioned, the original contract proposed research in mathematical analysis relating to electromagnetic waves, mechanical vibrations, and electrical networks. These original areas have broadened into other areas which, superficially, appear unrelated.

The research on wave theory was initiated by A. E. Heins. The initial goal of Heins and his collaborators was to derive exact solutions for electromagnetic waves under various conditions of diffraction. The importance of this work to science stems from the fact that there are very few exact solutions known in spite of great effort for many years to obtain such solutions.

The mathematical techniques employed by Heins involved deep results in complex variable theory. For example, diffraction problems were cast into the form of Wiener-Hopf integral equations. Moreover, as a by-product, the research showed how to extend the range of application of Wiener-Hopf theory.

When Heins returned from his sabbatical leave in Denmark, his research took on new direction. In particular, he and Professor Richard MacCamy collaborated on existence theory methods via complex variable techniques. After Heins left CMU,an active program in wave research was continued by Professors MacCamy, Mizel, and Greenberg. Their work has directly related to the AEMS work in the following two areas:

(1) Non-existence of solutions of nonlinear wave equations.

(2) Exponential decay of waves in resistive media.

At the beginning of the AEMS project the growing importance of the digital computer in applied mathematics was evident. However, a digital computer can only program a discrete system, but most problems in applied mathematics concern continuous systems. Hence the effectiveness of digital computers to applied mathematics depends on an analogy between continuous and discrete systems. Thus it seemed important to investigate the analogy.

Perhaps the main theme of the research has been developing this analogy between continuous and discrete systems. In particular, a discrete potential theory was developed by studying the difference equations analogous to the Laplace and Poisson equations. Likewise a discrete function theory was developed by studying difference equations analogous to the Cauchy-Riemann equations. In this way complex valued functions termed "discrete analytic" were introduced. Many of the Ph.D. theses treated such difference equation theories.

The product of discrete analytic functions is, unfortunately, not discrete analytic. However Charles Duris, in his thesis, developed a convolution product which preserves discrete analyticity. This result together with some work of Hans Lewy raised another question. Lewy had found a convolution product for solutions of certain partial differential equations which was again a solution. This led Joan Rohrer, in her thesis, to develop a convolution product which gave a generalization of the Lewy theory to all partial difference equations in two variables and having constant coefficients.

The above mentioned discrete potential theory may also be described as a network theory for a regular lattice network. This suggested that the analogy be extended to irregular networks. Such was carried out in a paper termed "Lumped and distributed networks". This paper gave algorithms for treating a conducting body as a lumped network. Of course the concept of a lumped network is not new, but this paper put it into a quantitative form by giving error estimates.

Some of the ideas in the above paper were rediscovered by engineers and made popular under the name "finite element method".

Another application of the continuous-discrete analogy concerned the concept of "extremal length" introduced by Ahlfors and Beurling to study conformal mapping. The analogy suggested a corresponding concept for networks. This led to a new network relation termed the "length-width inequality". Analogy is a two-edged sword -- so leaving the discrete system and returning to the continuous system suggested a length-width inequality for a conducting body. This was a new, non-trivial concept in classical potential theory. A proof was supplied by William R. Derrick.

Problems in elasticity and in viscous flow often reduce to properties of the biharmonic operator. This was a reason why much attention has been given to biharmonic equations in the AEMS project. The first result in this research area was a counterexample to Hadamard's conjecture that the Green's function of a clamped plate is positive. Similar investigations concerned

counterexamples to Szego's conjecture that the lowest mode of vibration of a clamped plate is positive.

Other studies of the biharmonic operator gave more constructive results. Thus one investigation extended the maximum principle for harmonic operators to biharmonic operators. Other studies extended the Schwarz reflection principle to the equations of elasticity.

In pure mathematics an unknown constant is often characterized by finding a sequence converging to it. In applied mathematics this solution is somewhat unsatisfactory. It is better to have dual converging sequences, one increasing and one decreasing. Then the desired constant can be estimated with a known bound on the error. This philosophy has motivated much of the AEMS research on estimates of eigenvalues and estimates of conductance. The error estimate has been termed a "duality inequality".

The theory of linear programming gives an instance of such a duality inequality. In a series of papers this duality theory was extended from finite dimensional space to Banach space. Moreover, application to approximation theory was developed.

A novel application of the duality theory was made to the optimum design of cooling fins. In a large class of such problems, it was found possible to give a closed form solution. This involved a new variational principle for second order differential equations.

Research on similar optimum design problems is continuing. In particular, the variational principle is being extended to

fourth order differential equations. This research is aimed at the design of mechanical systems having maximum strength for a given weight.

A second order differential equation may be replaced by an equivalent system of first order differential equations. This results in many unforseen advantages. The best known example of this is, of course, the replacement of the Laplace equation by the Cauchy-Riemann equations. Another example is the Klein-Gordon equation. Its replacement by first order equations leads to the algebra of Dirac matrices. Again, the Klein-Gordon equation can be replaced by the Proca equation, and this leads to the algebra of Duffin-Kemmer matrices.

A great deal of research on this project has been motivated by the idea of such a replacement. We have already mentioned that the Laplace difference equation may be replaced by a system of difference equations, the discrete Cauchy-Riemann equations. Robert Helmbold introduced and studied a semi-discrete Laplace equation in his thesis. Gary Kurowski then replaced this Laplace equation by semi-discrete Cauchy-Riemann equations and developed the consequence in a series of papers.

A pair of functions satisfying the classical Cauchy-Riemann equations are termed "conjugates". Under mild integrability conditions, it is known that a conjugate pair satisfies an integral equation termed a Hilbert transform. In Technical Report 28 it was found that there is a generalization of Cauchy-Riemann equations to three dimensions and a corresponding generalization of the Hilbert transform for these generalized conjugates.

In the classical theory it is well known that the product of conjugate harmonic functions is also harmonic. Elkins in Technical Report 22 considers the possibility of such a theorem in three dimensions. He finds simple harmonic functions which when multiplied by any other harmonic function (not a constant) gives a product which is not a harmonic function. This remarkable counterexample of Elkins kills a whole class of conceivable generalizations of analytic functions.

The Cauchy-Riemann equations are a linear homogeneous system. The novel paper "Yukawan Potential Theory" is a study of a corresponding non-homogeneous system. It results that there is a close correspondence with classical potential theory. In particular, there are analogs, of the Cauchy integral formula and the Taylor series.

The AEMS project has shown that electrical network theory is a source of many interesting questions in algebra. In particular, the research has concerned network determinants Grassmann algebra, Wang algebra, matroids, and matrices with polynomial elements. Some of the research involved the classical algebra as developed in the book of Van der Waerden. However, some of the algebraic questions arising were not of the classical type.

One of the more interesting algebraic questions encountered concerns the interconnection of n-port networks. Each such network is ascribed an n by n semi-definite matrix A, termed the impedance matrix. When two networks are connected in series, then the matrix for the system is simply $A + B$.

If the networks are connected in parallel then the system matrix is denoted by A : B and is termed the "parallel sum". This parallel sum has very many nice properties. For example, the parallel sum is associative. This is intuitively obvious from the network model; however, the proof is quite long.

Suppose two n-port networks are interconnected, with some of the ports being in parallel and some of the ports being in series. In such a hybrid connection the system matrix is denoted by A * B and is termed the "hybrid sum". This leads to an even richer algebraic theory than the parallel connection. For instance, there is an intimate connection with A. W. Tucker's theory of combinatorial equivalent matrices arising in linear programming.

Various other types of network interconnections can be defined, and this leads to associated sum operations. Work on this fruitful area of research is being continued by Anderson, Duffin, Trapp, and Morley.

VI. Abstracts of OOR Technical Reports (1951-1961)

Listed here are the technical reports prepared under OOR Contracts

DA-36-061-ORD-113

DA-36-061-ORD-277

DA-36-061-ORD-378

DA-36-061-ORD-490.

These contracts covered the period 1 July, 1951 to 31 August 1961.

TR 1. "Water Waves over a Channel of Infinite Depth"
T. R. Greene and A. E. Heins

The same mathematical methods are often common to apparently dissimilar physical situations. In this paper techniques are applied which Heins had previously developed for electromagnetic problems.

TR 2. "Coupling of Two Half Planes"
A. E. Heins and H. Feshbach

This concerns a problem in diffraction theory. The question was formulated as a singular integral equation, and this led to a solution in analytic form.

TR 3. "Discrete Potential Theory"
R. J. Duffin

This investigation concerns the flow of electricity, heat, or fluid in an infinite lattice network. The governing relations are the Laplace difference equation and the Poisson difference equation. Various questions were investigated in analogy with ordinary potential theory. These questions include an operational calculus, Green's function, the mean value theorem of Gauss, and Harnack's inequality. In the course of this investigation it was necessary to develop the theory of the asymptotic expansion of triple Fourier integrals; this work was presented in an appendix which is essentially a separate paper.

TR 4. "On Finding the Characteristic Equation of a Square Matrix"

E. Saibel and W. J. Berger

This concerns a rapid method of obtaining the secular equation. The work has application in finding the natural frequencies of vibration of mechanical systems.

TR 5. "On the Inversion of Continuant Matrices"
W. J. Berger and E. Saibel

This paper concerns finding the inverse of matrices which occur in the analysis of vibrations of certain mechanical systems. The results also apply to certain types of electrical filters.

TR 6. "Continuation of Biharmonic Functions by Reflection"
R. J. Duffin

Biharmonic functions are considered which satisfy boundary conditions on a flat plane. These boundary conditions are of the type met in the theory of elasticity. Simple formulae are found which give the analytic continuation of the biharmonic function across the plane. By the methods of inversion, analogous continuation formulae are found for spherical surfaces. The work in this paper is a generalization of the well-known Schwarz principle of reflection for harmonic functions.

TR 7. "Exponential Decay in Nonlinear Networks"
R. J. Duffin

The problem concerns the effect of damping on the free vibrations of mechanical or electrical systems. The systems treated are of a very general nonlinear character. For linear systems it is well-known that the vibrations decay between two exponential rates. The same behavior is proved for nonlinear systems.

TR 8. "Impossible Behavior of Nonlinear Networks"
R. J. Duffin

Electro-mechanical systems of a general, nonlinear character are considered. The resistors of the system, however, are restricted to be of a quasi-linear type. The inductors and capcitors are permitted to have hysteresis. It is shown that such a system cannot convert direct current to alternating current. In the proof appeal is made to the second law of thermodynamics.

TR 9. "Asymptotic Expansion of Double Fourier Transforms"
R. J. Duffin and D. H. Shaffer

This work is a continuation of No. 3. As was mentioned above, that report contains a discussion of the asymptotic expansion of triple Fourier integrals. Surprisingly enough, the two dimensional problem is more difficult. One of the difficulties concerns certain integrals involving Bessel functions. By the use of summability methods, these integrals were evaluated. In particular the asymptotic form of the two-dimensional Green's Function in discrete potential theory is found.

TR 10. "On the Diffraction of a Plane Wave by an Infinite Plane Grating"
George Baldwin and A. E. Heins

The case treated here concerns a plane wave normally incident upon an infinite plane grating. The spacing of the strips is the same as the width of the strips. The strips are assumed to be perfect conductors. The reflection and transmission coefficients have been found for a wide wave-length band, and the results are presented in a form suitable for numerical evaluation.

TR 11. "Formulae Relating Some Equivalent Networks"
R. J. Duffin and Elsa Keitzer

A linear electro-mechanical system is considered with two degrees of freedom. Of interest is the driving point impedance of the system. A network without transformers is found which has the same driving point impedance. It is somewhat remarkable that the formulae relating the parameters of the two networks are rational. The methods of proof make use of Howitt's paper on "Group Theory and the Electric Circuit" and the Bott-Duffin synthesis method.

TR 12. "Minimax Theory for Overdamped Networks"
R. J. Duffin

This report presents a new approach to the analysis of electrical or mechanical systems having large amounts of friction. It is found possible to extend the classical methods of Rayleigh, Ritz, and Courant concerning systems without friction. The condition that a system be "overdamped" is expressed by requiring that a certain biquadratic functional be positive. This condition is analogous to the requirement that the

roots of a quadratic equation be real. The decay constants (eigenvalues) may be defined directly by a minimax statement with regard to a certain functional. The effect of constraints on the system is then easily defined. The effect on the eigenvalues is similar to the effect in conservative systems.

TR 13. "Continuation of Biharmonic Functions by Reflection II"
R. J. Duffin

This report is an extension of No. 6. By the use of the mean value theorem for biharmonic functions, it was found possible to find a considerable simplification of the proof in the previous report. The new material added concerns steady viscous flow of an incompressible fluid. In such a flow the three components of velocity are biharmonic. It is shown that such a flow may always be defined in terms of three harmonic functions. The boundary condition at a wall is simply that the velocity vanish. Formulae are found to continue a flow across a plane wall.

TR 14. "Power Series Inversion of the Leontief Matrix"
W. J. Berger and E. Saibel

The problem treated concerns inversion of matrices of the Leontief type. F. V. Waugh proposed to employ the Neumann series to obtain the inverse in order to avoid accumulation of round-off errors. Here other types of power series inversion formulae are investigated. These give more rapid convergence than the series proposed by Waugh.

TR 15. "Elementary Operations Which Generate Network Matrices"
R. J. Duffin

The behavior of a network with n pairs of terminals is defined by an n in n matrix which matrix elements are rational functions of the frequency z. The following theorem is proved: Starting with identity matrix I and matrix zI as a basis, then any network matrix may be generated by the operation of addition, multiplication by constant matrices, and forming the inverse. This theorem gives a new method of synthesizing a network.

TR 16. "Approximate Solution of Differential Equations by a Variational Method"
R. J. Duffin and W. D. Serbyn

Of concern is the two-point boundary value problem for second order ordinary differential equations of self-adjoint type. A formula is found which furnishes

approximation which never exceeds the solution. This method may be used to approximate functions defined by integrals. As an example, analytic approximations to the error function are obtained.

TR 17. "The Excitation of a Perfectly Conducting Half-Plane by a Dipole"
A. E. Heins

Here it is shown that the problem can be reduced to the solution of two scalar problems shown in 1914 by H. M. MacDonald. The solution of the electromagnetic problems follows by differentiation from these scalar problems.

TR 18. "Basic Properties of Discrete Analytic Functions"
R. J. Duffin

Complex valued functions are considered which are defined at the points of the complex plane whose coordinates are integers. These points form a lattice which breaks up the plane into unit squares. A function is termed discrete analytic at one of these squares if the difference quotient across one diagonal is equal to the difference quotient across the other diagonal. Basic properties of such analytic functions are investigated which have well-known analogs in the classical continuous theory. These analogies include: conjugate harmonic functions, poles, contour integration, residues, Cauchy's integral, polynomials, expansions, multiplication, Liouville's theorem, Harnack's inequality, and Hilbert transforms.

TR 19. "The Scope and Limitation of the Methods of Wiener and Hopf"
A. E. Heins

This is a survey of the methods of Wiener and Hopf in the study of boundary value problems which arise in electromagnetic theory, acoustics, and hydrodynamics. The survey has been made with the idea of bringing out the general aspects of the methods.

TR 20. "Infinite Programs"
R. J. Duffin

Most of the development work on linear programming theory has been confined to finite programs. A program is termed finite if it involves only a finite number of variables and a finite number of constraint inequalities on these variables; otherwise, a program is termed infinite. In this paper the interesting duality theory which has been developed for finite linear

programs is extended in such a manner that it applies to infinite linear programs.

TR 21. "Picard's Theorem and Linear Differential Equations"
R. J. Duffin and Zeev Nehari

Linear Differential equations are considered whose coefficients are analytic functions of the independent variable. It is found that an equation of the nth order cannot have more than n essentially different solutions which are entire and free from zeros.

TR 22. "Orthogonal Harmonic Functions in Three Dimensions"
T. A. Elkins

In two dimensions it is easily shown by the use of the complex variable that given any non-constant harmonic function, a non-constant harmonic function can be found such that the product of the two is harmonic. It is shown by presenting a counterexample that this result does not extend to three dimensions.

TR 23. "Analytic Continuation in Elasticity"
R. J. Duffin

This paper concerns the equations of static equilibrium of an isotropic elastic body. Formulae are found which represent a state of stress in terms of three harmonic functions. These formulae are employed to develop an analog of the Schwarz reflection principle. This yields analytic continuation of the solutions of the equations across a plane boundary surface. The boundary conditions assumed at the plane surface are the vanishing of the surface displacements (or the vanishing of the surface forces).

TR 24. "Half-Plane Diffraction with Line Source Excitation"
Robert Gast

The methods of Wiener and Hopf are applied to the problems of a line source in the presence of a semi-infinite half plane. The method supplied, after some subtle analysis, the classical solution of MacDonald.

TR 25. "Diffraction by Two Parallel Half-Planes with Source Excitation"
Robert Gast

This work generalizes the original work of A. E. Heins (1948) who discussed the problem in the title under plane wave excitation. A line source is considered, instead of a plane wave and the problem is formulated as an integral equation of the Wiener-Hopf method. The

methods of Wiener and Hopf are used to provide the solution as well as the interesting physical parameters.

TR 26. "On Pseudo-Analytic Functions and Elliptic Equations"
Edwin H. Farr

The so-called similarity principle is used to derive certain properties of pseudo-analytic functions from the corresponding properties of analytic functions. These and other ideas are applied to extend the work of Lichtenstein on elliptic equations in two variables. In particular, it is shown that the gradient of the Green's function of simple connected domain does not vanish.

TR 27. "A Note on Poisson's Integral"
R. J. Duffin

Let a continuous function f be defined on the boundary of a convex region. Of concern in this note is a simple mean value formula which extends f to be a continuous function at all points of the region. If the region is a sphere it is shown that the extended function is harmonic in the sphere. It follows that for a sphere this mean value formula is equivalent to the well-known integral formula of Poisson.

TR 28. "Two-Dimensional Hilbert Transforms"
R. J. Duffin

Fourier transforms, Abel summability, Poisson's integral, Cauchy's integral, conjugate harmonic functions, and Hilbert transforms are topics which are known to have a natural relationship. In this paper each of these topics is considered in one higher dimension. For example, the notion of a pair of conjugate harmonic functions in three variables is introduced by a suitable generalization of the Cauchy-Riemann equations. It is found that the same relationships are maintained between the six topics after they are so generalized.

TR 29. "The Green's Function for Periodic Structures in Diffraction Theory with an Application to Parallel Plate Media"
A. E. Heins

A Green's function for periodic structures in diffraction theory is constructed. This function enables one to formulate the boundary value problem associated with such periodic structures as a wave-guide problem, rather than a free space problem. The mathematical difficulties encountered in the formulation of such problems in the past are now clarified.

TR 30. "Representation of Fourier Integrals as Sums III"
R. J. Duffin

In the two previous papers it was shown that under certain conditions the Fourier transform of a Mobius series is another Mobius series. In the present paper it is shown that the Fourier transform can be represented as a double Mobius series.

TR 31. "Nim-type Games"
D. C. Benson and E. W. Adams

Previous research has shown a certain phenomenon of periodicity associated with the game of Kayles and certain variations. In this paper it is found under what general conditions this periodicity holds.

TR 32. "Comments on the Treatment of Diffraction of Plane Waves"
A. E. Heins and S. Silver

This note clarifies some remarks made in the paper, "The Edge Conditions and Field Representation Theorem in the Theory of Electromagnetic Diffraction", which appeared in the Proceedings of the Cambridge Philosophical Society in 1955.

TR 33. "The Green's Function for Periodic Structures in Diffraction Theory with an Application to Parallel Plate Media II"
A. E. Heins

This report discusses the use of the Green's function constructed in No. 29 in the solution of boundary value problems associated with parallel plate media. The boundary conditions are of the Neumann type and accordingly require some modifications of the methods used in No. 29.

TR 34. "Semi-Discrete Potential Theory"
R. L. Helmbold

The equations of a relatively new variant of the difference-equation method of approximately solving problems in partial differential equations are investigated, the discussion being restricted to partial differential equations involving the Laplace operator. The variant consists, for the two dimensional case, in replacing one of the partial derivatives by a central difference and the other partial derivatives by an ordinary derivative, thus obtaining a system of ordinary differential equations to be solved in such a way as to satisfy certain boundary conditions. It is shown that the equations of this variant are equivalent to a variational principle, the analogue of the maximum principle holds, and an

operational calculus is developed. Also included are a Green's function and Green's identities, applications to the estimation of eigenvalues, and explicit solutions in rectangular regions. Generalizations to higher dimensions are indicated.

TR 35. "A Function Theoretic Solution of Certain Integral Equations I"
A. E. Heins and R. MacCamy

This report is concerned with the solution, by function theoretic methods, of a certain type of integral equation arising in the study of boundary problems for elliptic partial differential equations in two variables. The equation is converted into a functional relation for a multiple-valued analytic function defined on a Riemann surface and the solution of this relation is obtained. The method is carried out in detail for the integral equation of the Sommerfeld half-plane problem.

TR 36. "Difference Equations of Polyharmonic Type"
R. J. Duffin and E. P. Shelly

Of concern are partial difference equations with constant coefficients. Fourier methods are used to study growth rate theorems and properties of the fundamental solution. A study is made of polynomial solutions and continuation theorems. As is well known, there are differential operators such as $r \cdot \mathrm{grad}$ which convert harmonic functions into harmonic functions. Analogous operators are developed for the theory of discrete harmonic functions. By the use of such operators, it is found possible to give a complete evaluation of the fundamental solution of the Laplace difference equation in three dimensions. If f is harmonic, then $r^2 f$ is biharmonic. Relationships of this type are extended to the discrete case, and such relationships lead to a complete evaluation of the fundamental solution of the discrete biharmonic equation in two dimensions.

TR 37. "Distributed and Lumped Networks"
R. J. Duffin

The electrical system resulting when terminal surfaces are attached to a conducting body is termed a distributed network. An upper network is defined, in the paper, to be a lumped network such that the conductance between a pair of its terminals is not less than the conductance between the corresponding terminals of the distributed network. A lower network is defined likewise but gives lower bounds to conductance instead of upper bounds.

Systematic methods are developed for constructing upper and lower networks. These considerations give a practical numerical procedure for determining the properties of the electrical system. The same methods apply to thermal conductivity and other physical problems governed by Laplace's equation.

TR 38. "A Function Theoretic Solution of Certain Integral Equations II"
A. E. Heins and R. MacCamy

This report is a continuation of No. 32. Here one treats integral equations of the second kind with sum and difference kernels. The methods employed are extensions of those used previously.

TR 39. "An Analysis of the Wang Algebra of Networks"
R. J. Duffin

A Wang algebra is defined by the relations $x + x = 0$ and $x \cdot x = 0$ for each element x of the algebra. This algebra was devised to furnish a short-cut method for evaluating electric network discriminants. A discriminant is the determinant of the network equations of Kirchhoff expressed as a function of the resistances (or conductances) of the branches of the network. In this paper the Wang algebra is shown to be a special case of the better known algebra of Grassmann. Of basic importance in this connection are the Grassmann outer products whose coefficients are 1, -1, or 0. Such outer products are related to network discriminants and to certain subspaces of a vector space. This analysis reveals new properties of discriminants and delineates the possible range of application of Wang algebra.

TR 40. "An Inverse Sturm-Liouville Problem"
Allan D. Martin

The classical problem in differnetial equations is concerned with the quest for qualitative or quantitative information about its solutions. In a more recent type of problem certain information, such as the spectrum, is given and the problem is to determine the operators for which the given information is valid. It is a problem of this type which is considered in this paper. Our method is variational.

TR 41. "Axially Symmetric Solutions of Elliptic Differential Equations"
A. E. Heins and R. C. MacCamy

An investigation is made of the representation of solutions of the axially-symmetric elliptic equations. The representations are derived by exploiting the connection between such equations and singular initial value problems for hyperbolic equations. The result is a correspondence between solutions of the elliptic equations and functions of a complex variable. Certain boundary-value problems for the elliptic equations are solved explicitly or semi-explicitly with the aid of these representations.

TR 42. "Semi-Discrete Analytic Functions"
G. J. Kurowski

Of concern are functions of one continuous and one discrete variable defined on a semi-lattice, a uniformly spaced sequence of lines parallel to the real-axis. Functions of a complex variable on this semi-lattice whose real and imaginary parts satisfy a pair of equations obtained from the classic Cauchy-Riemann equations on replacing the y-derivative by either a symmetric or non-symmetric difference provide semi-discrete analogues for analytic functions. With path integration defined on the semi-lattice, analogues for Cauchy's integral theorem and formula are presented. The "derivative" and "indefinite integral" of a semi-discrete analytic function are also shown to be semi-discrete analytic. The family of semi-discrete analytic functions is not closed under the usual multiplication; consequently, a modified "multiplication" having this property is discussed. Appropriate analogues for the powers of z, and thus polynomials, are obtained. A method similar to analytic continuation is presented which enables suitable functions to be "continued" as semi-discrete analytic functions into a rectangular domain of the semi-lattice.

TR 43. "On the Scattering of Waves by a Disk"
A. E. Heins and R. C. MacCamy

This paper concerns the scattering of sound waves by a circular disk of so-called "soft" material. An explicit solution of this problem is possible by means of spheroidal harmonics but the resulting formulae are exceedingly complicated. An alternative method is to formulate the problem as an integral equation. A simple method of obtaining and treating the integral equation is given here.

TR 44. "Note on Polyharmonic Functions"
R. J. Duffin and Z. Nehari

If a function is both harmonic and positive then according to a classical inequality of Harnack the rate of growth of the function is limited. In this note a similar inequality is found for function which is polyharmonic and positive in a region. If the region is the whole plane then this inequality yields a simple proof of the theorem of Nicolesco to the effect that the function is a polynomial.

TR 45. "The Maximum Principle and Biharmonic Functions"
R. J. Duffin

This note concerns the maximum principle which applies to solutions of partial differential equations of elliptic type. This principle asserts that the maximum of a solution occurs on the boundary of a region. Consideration of the ratio of solutions of an elliptic equation shows that the ratio satisfies the same maximum principle. This result is then used to obtain a maximum principle relating to biharmonic functions. These maximum principles give inequalities which biharmonic functions must satisfy. The relations and concepts developed in this note have application in elasticity and in hydrodynamics.

TR 46. "An Exponomial Extrapolator
R. J. Duffin and Phillips Whidden

A linear combination of m exponential functions (exponomial) is fitted to a time series, such as daily observations. The fitting is carried out over all past time by weighted least squares with an exponential weight factor. The resulting minimizing function could be continued into the future. In particular tomorrow's predicted value is defined by this continuation. To obtain an explicit solution of the problem a formula is constructed which gives the predicted value as a linear combination of the last m observed values and the last m predicted values. The $2m$ coefficients of this formula are expressed as explicit rational functions of the m exponential bases. The extrapolating functions available with this method include polynomials, trigonometric polynomials, damped waves, etc. The particular class of extrapolating functions to be used for a given problem depends on the genesis of the data.

TR 47. "Nonsymmetric Projections in Hilbert Space"
V. J. Mizel and M. M. Rao

In this paper a characterization is given for the general (not necessarily symmetric) bounded linear idempotent operator, or projection, on Hilbert space, and several more or less direct consequences are deduced. These results are applied to the investigation of a specific type of projection problem and to a "weak" ordering of projection operators. A sketch of the role of these problems in statistical theory and a comparison of the results with previous studies completes the paper.

TR 48. "Orthogonal Saw-Tooth Functions"
R. J. Duffin

Harrington and Cell have defined an interesting set of square-wave functions. They have studied the properties of this set of functions and have shown them to be orthogonal and complete on L_2. This note carries out a similar study but starts with a saw-tooth wave rather than a square wave.

TR 49. "The Extremal Length of a Network"
R. J. Duffin

The extremal length of a plane region has been studied by Grotzsch, Beurling, and Ahlfors. Extremal length may be interpreted as electrical resistance and this suggests that the definition of extremal length be extended to networks. This extension is carried out in this paper with the aid of linear programming theory. This leads to a new definition for the resistance of a network and to new network inequalities.

TR 50. "The Reciprocal of a Fourier Series"
R. J. Duffin

Given the Fourier coefficients of two functions relative to an orthonormal sequence a procedure is developed for finding the Fourier coefficients of the ratio of the two functions. This procedure is an extension of a method given by Edrei and Szego for a more special problem.

TR 51. "Pseudo-Hamiltonian Mechanics"
R. J. Duffin

Hamilton's canonical equations form a set of 2n ordinary differential equations which can be used to describe

the motion of a conservative mechanical system with n degrees of freedom. This paper is a study of a generalization in which an additional parameter is introduced in Hamilton's equations. The new set of equations can be used to describe the motion of systems which dissipate energy. In particular, these equations may be applied to electrical networks.

TR 52. "On Exterior Boundary Values in Linear Elasticity"
R. J. Duffin and W. Noll

Of concern is an infinite elastic medium. This paper resolves the question of the proper boundary conditions to be assigned at infinity. The results also apply to flow in an infinite viscous fluid.

VII. WADC Technical Reports (1951-1956)

Listed here are technical reports prepared under contract with the Wright Air Development Center, Dayton, Ohio. Only the reports pertaining to the AEMS project are listed.

1. "Vibration of a Ring-Shaped Plate", TR 16, (July 1952), Contract AF-33(616)-1

   R. J. Duffin and D. H. Shaffer

   It is shown that the lowest frequency of vibration of a clamped plate may have a nodal line. This is a counter-example to a conjection of Szego. This report was presented to the AMS. It is published only in abstract form in the Bulletin of the American Mathematical Society, 1953.

2. "Natural Vibrations of Cantilevered Triangular Plates I", TR 22, (September 1952), Contract AF-33(616)-1

   R. J. Duffin, P. N. Gustafson and W. H. Warner

   This was the first of a series of studies on a model of the delta wing.

3. "An Experimental Study of Natural Vibrations of Cantilevered Triangluar Plates", TR 26, (October 1952), Contract AF-33(616)-1

   P. N. Gustafson, W. Stokey, and C. Zorowski

4. "The Effect of Small Constraints on Natural Vibrations", TR 28, (September 1953), Contract AF-33(616)-277

   R. J. Duffin and A. Schild

   This paper gave an algorithm which made Rayleigh's principle, a qualitative statement, into a quantitative relation.

5. "Statistics of Particle Measurement and of Particle Growth", TR 32, (July 1954), Contract AF-33(616)-277

   R. J. Duffin, R. A. Meussner, and F. N. Rhines

   This concerns certain problems in metallurgy. The paper has not been published but the relations developed have received continued application in latter investigations of Rhines.

6. "Natural Vibrations of Cantilevered Triangular Plates II", WADC TR 54-358, (July 1954), Contract AF-33(616)-277.

   R. J. Duffin and P. N. Gustafson

7. "Energy Solution for Natural Vibrations of Cantilevered Triangular Plates" (June 1956).

   R. J. Duffin, P. N. Gustafson, and W. A. Warner

Of course the "triangular plates" mentioned in these reports were models of the delta wing. At that time, delta wings were a new development in aviation.

VIII. OOR Technical Reports at Duke University (1958-1960)

Listed here are technical notes prepared by R. J. Duffin at Duke University under the Office of Ordnance Research, Contract DA-31-124-ORD.

1. "Vibrations of a Rotating Wire", (November 1958).
2. "Extrapolation with Polynomials", (December 1958), with T. W. Schmidt.
3. "Differential Equations for the Stress in Cylinders with Reinforcing Rings", (January 1959).
4. "A Simple Formula for Prediction and Automatic Scrutation", (February 1959), with T. W. Schmidt.
5. "On the Capacitance Network of a System of Three Conductors I", (March 1959).
6. "The Rayleigh-Ritz Method for Dissipative or Gyroscopic Systems", (May 1959).
7. "On the Capacitance Network of a System of Three Conductors III", (May 1959).
8. "Discounted Least Squares", (June 1959). with T. W. Schmidt.
9. "On the Probability of Bomb Fragments Hitting a Target", (July 1959).
10. "A Theory of Ring-Stiffened Cylinders", (September 1959).
11. "An Extrapolator and Scrutator", (November 1959), with T. W. Schmidt.
12. "On the General Solution of the Three-Dimensional Viscoelastic Navier Equation for Commutative Systems", (April 1960), with G. J. Kurowski.
13. "On the Tip-Off Problem in Ramp-Launched Craft", (April 1960), with G. J. Kurowski.
14. "The Maximum Principle in Hydrodynamics", (August 1960).

IX. History of the AEMS Project (1960-1970)

The second ten years of the AEMS project began on 1 September 1961 under Research Grant DA-AROD-31-124-G78. The support was now from the Army Research Office in Durham, which replaced the Office of Ordinance Research. Albert Heins had accepted a position at the University of Michigan, so other researchers at Carnegie have replaced him in the AEMS project.

The first AROD research grant was followed by Grants DA-AROD-31-124, G78, G602, G680, G951, G17. The list of research publications during the second ten years is listed in Appendix H.

Duffin has often served as a consultant to the Westinghouse Research Laboratories. Most problems coming to his attention in that way were of a standard nature, but some promised novelty. For example, a question concerning the optimum design of a cooling fin led to a new class of problems in the calculus of variations. Another question concerning levitation led to a proof that dielectric constant of a substance can never be less than unity. Other Westinghouse problems are listed in Appendix G.

Clarence Zener, as director of the Westinghouse Research Laboratories, recognized a need for a systematic method in the optimization of engineering designs. Zener, Duffin and Peterson developed a mathematical framework sufficiently flexible to solve many engineering problems. They wrote a book on this new method of Geometric Programming. Geometric

programming is a generalization of linear programming; linear functions are replaced by polynomial functions. This optimization method is now being further developed by many authors both here and abroad.

Professor Dov Hazony of the Department of Electrical Engineering at Case-Western Reserve University paid a visit to Carnegie-Mellon University to ask questions concerning the Bott-Duffin synthesis method. This visit started a fruitful collaboration involving Duffin, Hazony, and several research assistants. Many trips between Cleveland and Pittsburgh were made. A result of this collaboration were two new methods for the synthesis of n-port networks. Moreover, the proofs required development of new concepts concerning matrices whose matrix elements are rational functions of a complex variable.

In the academic year 1967-68, Duffin was a visiting professor; the first semester at Stony Brook and the second semester at Texas A and M. He was accompanied by research assistant William N. Anderson. During this year their main research effort was developing a new matrix operation termed "parallel addition". This operation was suggested by network theory, and it turned out to be a new concept in pure mathematics.

In 1968, Clarence Zener accepted a professorship at Carnegie-Mellon University. This gave an opportunity for Duffin and Zener to renew collaboration on geometric programming. In one paper they introduced a new concept in chemical thermo-

dynamics termed the "anti-entropy function". In another paper they related the Darwin-Fowler method of statistical mechanics to geometric programming.

X. Westinghouse Technical Reports (1958-1969)

Reports related to the AEMS project prepared by R. J. Duffin for the Westinghouse Research Laboratory:

1. "Solution to a Problem in Locked Diffusion", (June 1958), with R. H. Boyer.

2. "The Differential Equation of the Thermocouple", (February 1959).

3. "Constrained Minima Treated by Geometric Means", (March 1964), with E. L. Peterson.

4. "Mathematical Analysis for the Resonant Gate Transistor I,II", (December 1966).

5. "Coordination of Large System Design Efforts Using Geometric Programming", (November 1966), with A. Federowicz.

6. "Stability of Integration Formulae for Simultaneous Differential Equations", (August 1968), with B. Swanson.

7. "Numerical Solutions of Static Field Problems Based on Integral Equations", (November 1969), with J. H. McWhirter.

XI. Publications (1960-1971)

AEMS research papers appearing from 1960

60-1 "The Rayleigh-Ritz method for dissipative or gyroscopic systems", Quarterly of Appl. Math. 18, 215-221.
R. J. Duffin

60-2 "Asymptotic expansion of double Fourier transforms", Duke Math. Jour. 27, 581-596.
R. J. Duffin and D. H. Shaffer

60-3 "Simple formula for prediction and automatic scrutation", Amer. Rocket Soc. Jour. 30, 364-365.
R. J. Duffin and T. W. Schmidt

60-4 "Discounted least squares", Proceedings of the Fourth Ordnance Conference on Operations Research, Army Research Office, Durham
R. J. Duffin

60-5 "An extrapolator and scrutator", Jour. of Math. Anal. and Applications 1, 215-227.
R. J. Duffin and T. W. Schmidt.

61-1 "Note on polyharmonic functions". Proc. Amer. Math. Soc. 12, 110-115.
R. J. Duffin and Z. Nehari

61-2 "A theory of ring-stiffened cylinders", Jour. of the Aerospace Sciences 28, 702-709.
R. J. Duffin

61-3 "The maximum principle and biharmonic functions", Jour. of Math. Anal. and Applications 3, 399-405.
R. J. Duffin

61-4 "An exponomial extrapolator", Jour. of Math. Anal. and Applications 3, 526-536.
R. J. Duffin and Phillips Whidden

61-5 "Orthogonal saw tooth functions", Duke Math. Jour. 28, 559-562.
R. J. Duffin

61-6 "Programmes in paired spaces", Can. Jour. Math. 13, 221-238.
K. S. Kretschmer

62-1 "Dual programs and minimum cost", Jour. Soc. Ind. Appl. Math. 10, 119-123.
R. J. Duffin

62-2 "The extremal length of a network", Jour. Math. Anal. and Applications 5, 200-215.
R. J. Duffin

62-3 "Cost minimization problems treated by geometric means", Operations Research 10, 668-675.
R. J. Duffin

62-4 "Pseudo-Hamiltonian mechanics", Archive for Rat. Mech. and Anal. 9, 309-318.
R. J. Duffin

62-5 "The reciprocal of a Fourier series", Proc. Amer. Math. Soc. 13, 965-970.
R. J. Duffin

63-1 "The degree of a rational matrix function", Jour. Soc. Ind. Appl. Math. 11, 645-658.
R. J. Duffin and D. Hazony

63-2 "Free suspension and Earnshaw's theorem", Archive for Rat. Mech. and Anal. 14, 261-263.
R. J. Duffin

63-3 "Richard's Transformations which generate network matrices", Chapter 14, Elements of Network Synthesis by D. Hazony, Reinhold.
R. J. Duffin, D. Hazony, E. K. Boyce, and H. Nain

63-4 "Semi-discrete analytic functions", Trans. Amer. Math. Soc. 106, 1-18.
G. J. Kurowski

63-5 "Averagings and quadratic equations in operators", CIT Report 9.
V. J. Mizel and M. M. Rao

64-1 "Chrystal's theorem on differential equation systems", Jour. of Math. Anal. and Applications 8, 325-331.
R. J. Duffin

64-2 "A convolution product for discrete function theory", Duke Math. Jour. 31. 199-220.
R. J. Duffin and C. S. Duris

64-3 "Optimization of engineering problems", Westinghouse Engineer 24, 154-160.
R. J. Duffin and C. Zener

64-4 "Discrete analytic continuation of solutions of difference equations", Jour. of Math. Anal. and Applications 9, 252-267.
R. J. Duffin and C. S. Duris

64-5 "The potential energy of an electric charge", Archive for Rat. Mech. and Anal. 15, 305-310.
R. J. Duffin

64-6 "A duality theory for infinite linear programs", Ph.D. Thesis.
L. A. Karlovitz

64-7 "Asymptotic expansion of multiple Fourier integrals", Ph.D. Thesis.
M. Mangad

64-8 "Inequalities concerning Jacobi polynomials", Ph.D. Thesis.
M. L. Patrick

64-9 "Some properties of complex valued lattice functions", Ph.D. Thesis.
E. L. Peterson

64-10 "Network models for conductance", Ph.D. Thesis.
T. A. Porsching

65-1 "Topology of series-parallel networks", Jour. of Math. Anal. and Applications 10, 303-318.
R. J. Duffin

65-2 "Instability, uniqueness, and nonexistence theorems for the equation $u_t = u_{xx} - u_{xtx}$ on a strip", Archive for Rat. Mech. and Anal. 19, 100-116.
R. J. Duffin, B. D. Coleman and V. J. Mizel

65-3 "An infinite linear program with a duality gap", Management Science 12, 122-134.
R. J. Duffin and L. A. Karlovitz

65-4 "Difference equations on non-rectangular lattices in three-dimensional space". Ph.D. Thesis.
Mary J. Winter

65-5 "The D-transform in discrete function theory", Ph.D. Thesis.
Alan Washburn

66-1 "Network synthesis through hybrid matrices", Jour. Soc. Ind. Appl. Math. 14, 413-490.
R. J. Duffin, D. Hazony and N. Morrison

66-2 "Duality theory for geometric programming", Jour. Soc. Ind. Appl. Math. 18, 1307-1349.
R. J. Duffin and E. L. Peterson

66-3 "Bounds for the conductance of a leaky plate via network models", Proc. Symp. on Generalized Networks, Polytechnic Institute of Brooklyn.
R. J. Duffin and T. A. Porsching

66-4 "The gyration operator in network theory", Proc. Symp. on Generalized Networks, Polytechnic Institute of Brooklyn.
R. J. Duffin, D. Hazony and N. Morrison

66-5 "Complex space-time and classical field theory", Jour. of Math. Phys. 7, 45-51.
A. J. Das

66-6 "Complex space-time and quantum theory of free fields", Jour. of Math. Phys. 7, 52-60.
A. J. Das

66-7 "Complex space-time and geometrization of electromagnetism", Jour. of Math. Phys. 7, 61-63.
A. J. Das

67-1 "An orthogonality theorem of Dines related to moment, problems and linear programming", Jour. Comb. Theory 2, 1-26.
R. J. Duffin

67-2 Geometric Programming, John Wiley and Sons, Inc.
R. J. Duffin, E. L. Peterson and C. Zener

67-3 "Extrapolating time series by discounted least squares", Jour. of Math. Anal. and Appl. 20, 325-341.
R. J. Duffin

67-4 "The potential energy of an electric charge is a superharmonic function", Arch. Rat. Mech. and Anal. 25, 156-158.
R. J. Duffin and A. Schild

67-5 "Optimization problems related to heat flow", Ph.D. Thesis.
D. K. McLain

67-6 "Generalization of discrete analytic functions", Ph.D. Thesis.
Joan Rohrer

67-7 "Asymptotic expansions of Fourier transforms and discrete polyharmonic Green's functions", Pac. Jour. of Math. 20, 85-98.
Moshe Mangad

67-8 "On conjugate discrete harmonic functions", National Bureau of Standards Jour. of Research <u>71B</u>, 105-110.
E. L. Peterson

68-1 "Estimating Dirichlet's integral and electrical resistance for systems which are not self-adjoint", Arch. Rat. Mech. and Anal. <u>30</u>, 90-101.
R. J. Duffin

68-2 "Formulation of linear programs in analysis I: Approximation theory", SIAM Jour. of Appl. Math. <u>16</u>, 662-675.
R. J. Duffin and L. A. Karlovitz

68-3 "A convolution product for the solutions of partial difference equations", Duke Math. Jour. <u>35</u>, 683-698.
R. J. Duffin and Joan Rohrer

68-4 "Optimum shape of a cooling fin on a convex cylinder", Jour. Math. and Mech. <u>17</u>, 769-784.
R. J. Duffin and D. K. McLain

68-5 "Optimum heat transfer and network programming", Jour. Math. and Mech. <u>17</u>, 759-768.
R. J. Duffin

68-6 "The discrete analog of a class of entire functions", Jour. Math. Anal. and Appl. <u>21</u>, 619-642.
R. J. Duffin and E. L. Peterson

68-7 "Stability of systems with nonlinear damping", Jour. Math. Anal. and Appl. <u>23</u>, 428-439.
R. J. Duffin

68-8 "Potential theory on a rhombic lattice", Jour. of Combinatorial Theory <u>5</u>, 258-272.
R. J. Duffin

68-9 "Optimization of a finite system of related differences", M.S. Thesis.
John Lackmann

68-10 "Optimization of engineering design by geometric programming", ARO(D) Report 68-1, Transactions of the Thirteenth Conference of Army Mathematicians.
R. J. Duffin and C. Zener

68-11 "Series and parallel addition of operators", Ph.D. Thesis.
W. N. Anderson, Jr.

68-12 "Algorithms and recursive functions", Ph.D. Thesis.
Donald Taranto

69-1 "The influence of Poisson's ratio on the vibrational spectrum", SIAM Jour. Appl. Math. 17.
R. J. Duffin

69-2 "Algorithms for classical stability problems", SIAM Review 11, 196-213.
R. J. Duffin

69-3 "Series and parallel addition of matrices", Jour. Math. Anal. and Appl. 26, 576-594.
R. J. Duffin and W. N. Anderson, Jr.

69-4 "Geometric programming, chemical equilibrium, and the antientropy function", Proc. Nat. Acad. Sciences 63, 629-639.
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69-5 "Network models", Proceedings of the Symp. on Mathematical Aspects of Electrical Network Theory, American Mathematical Society.
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69-6 "A generalization of discrete analytic and harmonic functions", Jour. Math. Anal. and Appl. 25, 628-652.
R. J. Duffin and Joan Rohrer Hundhausen

69-7 "The Schwarzian derivative and disconjugacy of n-th order linear differential equations", Can. Jorn. Math. 21, 235-249.
Meira Lavie

69-8 "Some function-theoretic aspects of disconjugacy of linear differential equations", Trans. Amer. Math. Soc. 143, 153-171.
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69-9 "Analytic iterations on Riemann surfaces", Bull. Aus. Math. Soc. 1, 183-194.
Meira Lavie

70-1 "Geometric programming and the Darwin-Fowler method in statistical mechanics", Jour. of Phys. Chemistry 74, 2419-2923.
R. J. Duffin and C. Zener

70-2 "Disconjugacy of linear differential equations in the complex domain", Pac. Jour. Math. 32, 435-457.
Meira Lavie

70-3 "On disconjugacy and interpolation in the complex plane", Jour. Math. Anal. Appl. 32, 246-263.
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70-4 "A treatise on realcompactness", Ph.D. Thesis.
Maurice Weir

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George Trapp

70-6 "Linearizing geometric programming", SIAM Review 12, 211-227.
R. J. Duffin

70-7 "Equipartition of energy in wave motion", Jour. Math. Anal. and Appl. 32, 286-291.
R. J. Duffin

71-1 "Geometric programming with signomials", Jour. Optimization Theory.
R. J. Duffin and E. L. Peterson

71-2 "Hybrid addition of matrices - A network theory concept", Jour. of Applicable Analysis.
R. J. Duffin and George Trapp

71-3 "Geometric programs treated with slack variables", Jour. of Applicable Analysis.
R. J. Duffin and E. L. Peterson

71-4 "Yukawan Potential Theory", Jour. Math. Anal. and Appl.
R. J. Duffin

71-5 "Vibration of a beaded string analyzed topologically", Jour. of Applicable Analysis.
R. J. Duffin

71-6 "Reversed geometric programs treated by harmonic means".
R. J. Duffin and E. L. Peterson

71-7 "On subobjects in categories", Archiv der Mathematik.
Oswald Wyler

XII. Research Highlights (1970-1982)

A lively conference, Constructive Approaches to Mathematical Models, was held at Carnegie-Mellon University, July 10-14, 1978. This conference was in honor of R. J. Duffin. There were over fifty papers presented by mathemeticians and engineers. The papers were in the area of Graphs and Networks, Mathematical Programming, Differential Equations and Mathematical Models. The proceedings of the conference was published by Academic Press in 1979.

There follows a list of papers published since 1970. The topics studied were quite varied so rather than attempting to summarize the total work, an abstract of each paper is given. There is, however, an underlying theme in that most of the papers involve a mathematical model of a physical or economic situation.

1. "Network models"
Mathematical Aspects of Electrical Network Theory, SIAM-AMS Proceedings 3 (1971), 65-91.

The steady flow of electrical current through a network of conductors has served as a suggestive model for a variety of mathematical theories. This paper describes electrical models related to the following theories; series-parallel graphs, parallel addition of matrices, lattice theory, generalized inverses, Grassmann algebra. Wang algebra, matroids, extremal length. Rayleigh's reciprocal relation and the width-length inequality.

2. "Geometric programming and the Darwin-Fowler method in statistical mechanics"
Journal of Physical Chemistry 74 (1970), 2419-2423 (with C. Zener)

This paper concerns the classical problem of chemical equilibrium as formulated in the language of geometric programming. Thus the equilibrium state at constant temperature and volume is characterized by the duality principle, minimum F =

maximum F^*. Here F is the Helmholtz function for free energy and F^* is a new function termed the anti-Helmholtz function. The minimization of F is constrained by the mass balance equations. However the maximization of F^* is unconstrained. Hence this gives a simplified practical procedure for calculating equilibrium concentration. The chemical equilibrium can also be analyzed by statistical mechanics. Comparing the two methods brings to light an intimate relationship between geometric programming and Darwin-Fowler statistics.

3. "Equipartition of energy in wave motion"
Journal of Mathematical Analysis and Applications 32(1970), 386-391.

Of concern are solutions of the classical wave equation in three-dimensions. It is shown that if a solution has compact support then after a finite time, the kinetic energy of the wave is constant and equals the potential energy. The proof employs the Paley-Wiener theorem of Fourier analysis.

4. "Duality inequalities of mathematics and science"
Nonlinear Programming, Edited by J. B. Rosen, O. L. Mangasarian, and K. Ritter, Academic Press, New York, 1970, 401-423.

The problem of minimizing a scalar functional $u(x)$ under a set of constraints S in the vector variable x is termed a program. It often results that there is an associated program of maximizing a scalar functional $v(y)$ under a set of constraints T on the vector y. These programs are termed dual if it can be shown that the functional $u(x)$ exceeds the functional $v(y)$ Then there exists a constant M such that

$$\underset{x \in S}{u(x)} \geq M \geq \underset{y \in T}{v(y)}$$

The virtue of this duality inequality is that it permits estimating M with a known bound on the error. Inequalities of this form appear in various areas of mathematics, science, engineering, and economics. This paper points out several such duality inequalities and their interrelationships.

5. "Yukawan potential theory"
Journal of Mathematical Analysis and Applications 35(1971), 70-13.

This paper concerns the Yukawa equation $\Delta u = \mu^2 u$ where μ is a real constant. Given a solution $u(x,y)$ of this equation then there is a conjugate function $v(x,y)$

satisfying the same equation and related to $u(x,y)$ by a generalization of the Cauchy-Riemann equations. This gives rise to interesting analogies with logarithmic potential theory and with complex function theory. In particular there are generalizations of holomorphic functions, Taylor series, Cauchy's formula, and Rouche's theorem. The resulting formulae contain Bessel functions instead of the logarithmic functions which appear in the classical theory. However, as $\mu \to 0$ the formulae revert to the classical case. A convolution product for generalized holomorphic functions is shown to produce another generalized holomorphic function.

6. "Vibration of a beaded string analyzed topologically" Applicable Analysis 56(1974), 287-293.

Of concern are the transverse vibrations of a finite string of beads. It is shown that a periodic vibration can result when the beads are released from an initial configuration. Moreover a norm on the initial configuration can be given a prescribed value. The proof uses the Brouwer fixed point theorem.

7. "Geometric programming with signomials" Journal of Optimization Theory 11(1973), 3-35 (with E. L. Peterson)

The difference of two "posynomials" (namely, polynomials with arbitrary real exponents, but positive coefficients and positive independent variables) is termed a "signomial".

Each signomial program (in which a signomial is to be either minimized or maximized subject to signomial constraints) is transformed into an equivalent posynomial program in which a posynomial is to be minimized subject only to inequality posynomial constraints. The resulting class of posynomial programs is significantly larger than the class of (prototype) posynomial programs in which a posynomial is to be minimized subject only to upper-bound inequality posynomial constraints. However, much of the (prototype) geometric programming theory is generalized by studying the "equilibrium solutions" to the "reversed geometric programs" in this larger class.

8. "Hybrid addition of matrices - A network theory concept" Applicable Analysis 2(1972), 241-254 (with G. E. Trapp).

The parallel connection of networks suggested the concept of parallel addition of matrices to Anderson and Duffin. The hybrid connection of networks also suggests a matrix operation. Using the Kirchhoff current and voltage equations, a new operation, hybrid addition, is defined for the set of Hermitian semidefinite matrices. This operation is an Hermitian semi-

definite order preserving semigroup operation. Hybrid addition is closely related to the work of Anderson on "shorted operators" and to the gyration operation of linear programming and network synthesis.

9. "Geometric programs treated with slack variables" Applicable Analysis 2(1972), 255-267 (with E. L. Peterson).

Kochenberger and Woolsey have introduced slack variables into the constraints of a geometric program and have added their reciprocals to the objective function. They find this augmented program advantageous for numerical minimization. In this paper the augmented program is used to give a relatively simple proof of the "refined duality theory" of geometric programming. This proof also shows that the optimal solutions for the augmented program converge to the (desired) optimal solutions for the original program.

10. "Reversed geometric programs treated by harmonic means" Indiana University Mathematics Journal 22(1972), 531-550 (with E. L. Peterson).

A "posynomial" is a (generalized) polynomial with arbitrary real exponents, but positive coefficients and positive independent variables. Each posynomial program in which a posynomial is to be minimized subject to only inequality posynomial constraints is termed a "reversed geometric program".

The study of each reversed geometric program is reduced to the study of a corresponding family of approximating (prototype) "geometric programs" (namely, posynomial programs in which a posynomial is to be minimized subject to only upper-bound inequality posynomial constraints). This reduction comes from using the classical arithmetic-harmonic mean inequality to "invert" each lower-bound inequality constraint into an equivalent "robust" family of "conservatively approximating" upper-bound inequality constraints. The resulting families of approximating geometric programs are then studied with the aid of the techniques of (prototype) geometric programming.

11. "Adjacency matrix concepts for the analysis of the interconnection of networks" Journal Franklin Inst. 298(1974), 9-27 (with W. N. Anderson, Jr. and G. E. Trapp).

A mathematical theory is developed for an imagined device termed a "junctor". A junctor could be used to interconnect

two n-terminal networks giving rise to another n-terminal network. Actually a junctor is itself a simple network with three banks of n terminals internally connected in some fashion by perfectly conducting wires. Incidence matrices are formulated to analyze various junctors and their current flows. The main problem treated concerns conditions which ensure that the junctor operation is associative.

12. "Network models for maximization of heat transfer under weight constraints"
Journal of Networks 2(1972), 71-48 (with S. Bhargava).

Of concern is a network in which the conductance of certain branches are variable. The problem posed is the maximization of the joint conductance subject to a bound on the ℓ_ρ norm of the variable conductances. It is shown that at an optimum state the conductance of a variable branch is proportional to the $2/(\rho + 1)$ power of the current through the branch. This relation together with a dual variational principle leads to a "duality inequality" giving sharp upper and lower estimates of the maximum joint conductance. Such a network serves as a discrete model for a cooling fin subject to a weight limitation. Thus the model shows what analogous properties should hold for the cooling fin.

13. "Dual extremum principles relating to cooling fins"
Quarterly of Applied Mathematics 31(1973), 27-41 (with S. Bhargava).

Under consideration is a differential equation $(pu')' = qu$ of the Sturm-Liouville type where the function $q(x) > 0$ is given. The problem is to find a function $p(x) > 0$ in $0 < x < b$, a constant b and a solution $u(x)$ of the corresponding differential equation such that the energy functional

$$\int_0^b [p(u')^2 + qu^2]\,dx$$

is maximized when $p(x)$ is subject to the constraint $\int_0^b p^\rho dx \leq K$ and u is subject to the boundary conditions $u = 1$ at $x = 0$ and $p\frac{du}{dx} = 0$ at $x = b$. Here $K > 0$ and $\rho \geq 1$ are constants. A pair of dual extremum principles is found to give sharp upper and lower estimates of the maximum value of the energy functional.

14. "Dual extremum principles relating to optimum beam design" Archive for Rat. Mech. and Analysis 50(1973), 314-330 (with S. Bhargava).

Of concern is a cantilever beam resting on an elastic foundation and supporting a load at the free end. The beam is of rectangular cross section and of constant height but variable width. It is required to taper the beam for maximum strength. This means that the beam is to support a maximum vertical load W at the free end when the free end is given unit deflection. The constraint is that the weight of the beam should not exceed a given bound K. It is shown that the optimum taper should be so chosen that the curvature of the beam is constant. This yields the solution of the problem in terms of explicit formulas. For more general constraints, an inequality is found which gives upper and lower bounds for the maximum load W even though explicit formulas are not available.

15. "On the nonlinear method of Wilkins for cooling fin optimization" SIAM J. Appl. Math. 24(1973), 441-448 (with S. Bhargava).

Of concern is the nonlinear differential equation equation $(k(u)p(x)u')' = p^{\eta}(x)Q(u) \quad 0 \leq x \leq b$ subject to the boundary conditions: $u = 1$ at $x = b$ and (kpu') takes the values $y_o > 0$ and 0 at $x = b$ and $x = 0$ respectively. Here $0 \leq \eta < 1$ is a given constant and k and Q are known functions and the question posed is to find a positive constant $b > 0$, a function $p(x) > 0$ in $0 < x \leq b$ and a solution u of the differential equation such that the norm $(\int_0^b p^{\rho}dx)^{1/\rho}$, $p \geq 1$ is minimized. A special transformation of variables together with Hölder's inequality leads to the solution in terms of explicit quadrature formulas.

16. "Matrix operations induced by network connections" SIAM J. Control 13(1975), 446-461 (with W. N. Anderson and G. E. Trapp).

In this paper a confluence is defined as a subspace of 3n-dimensional space having an indefinite inner product with signature (n,n,-n). Physically a confluence represents the vector space of all currents allowed by a given network interconnection. The space of voltages is then the orthogonal

complementary confluence.

17. "Convex analysis treated by linear programming" Journal of Mathematical Programming 4(1973), 125-143.

The theme of this paper is the application of linear analysis to simplify and extend convex analysis. The central problem treated is the standard concex program - minimize a convex function subject to inequality constraints on other convex functions. The present approach uses the support planes of the constraint region to transform the convex program into an equivalent linear program. Then the duality theory of infinite linear programming shows how to construct a new dual program of bilinear type. When this dual program is transformed back into the convex function formulation it concerns the minimax of an unconstrained Lagrange function. This result is somewhat similar to the Kuhn-Tucker theorem. However, no constraint qualifications are needed and yet perfect duality maintains between the primal and dual programs.

18. "The proximity of (algebraic) geometric programming to linear programming" Mathematical Programming 3(1972), 250-253, (with E. L. Peterson).

Geometric programming with (posy)monomials is known to be synonomous with linear programming. This note reduces algebraic programming to geometric programming with (posy)binomials.

19. "Tripartite graphs to analyze the interconnection of networks" Graph Theory and Applications, Edited by Y. Alavi, D. R. Lick, and A. T. White, 1972, 7-12.

In electrical network theory, many properties of connected networks are determined primarily by the connection and not the particular components that are connected. In this paper, we begin by viewing the interconnection of networks as a graph defined on three sets of vertices. By considering the networks as graphs, we are able to employ the concepts of adjacency matrices. We obtain results concerning interconnected graphs that are independent of our electrical network model.

20. "Nonuniformly elliptic equations: positivity of weak solutions" Bulletin of the American Mathematical Society 79(1973), 496-499 (with C. V. Coffman and V. J. Mizel).

This is a study of a class of degenerate second order elliptic operators. It is shown that: (i) the first eigenfunction and the Green's function are strictly positive and (ii) a non-negative eigenfunction necessarily minimizes the Rayleigh quotient on the appropriate function space. The motivation for this study was a certain nonlinear eigenvalue problem.

21. "Positivity of weak solutions of non-uniformly elliptic equations" Annali di Matematica pura ed applicata CIV(1975), 209-238, (with C. V. Coffman and V. J. Mizel).

Let A be a symmetric $N \times N$ real-matrix-valued function on a connected region Ω in R^n with A positive definite a.e. and A, A^{-1} locally integrable. Let b and c be locally integrable, non-negative, real-valued functions on Ω, with with c positive, a.e. Put $a(u,v) = \int_\Omega ((A\nabla u, \nabla v) + buv)dx$. We consider the boundary value problem $a(u,v) = \int_\Omega fvcdx$, for all $v \in C_0^\infty(\Omega)$. and the eigenvalue problem $a(u,v) = \lambda \int_\Omega uvcdx$, for all $v \in C_0^\infty(\Omega)$. Positivity of the solution operator for the boundary value problem, as well as positivity of the dominant eigenfunction (if there is one) and simplicity of the corresponding eigenvalue are proved to hold in this context.

22. "Parallel subtraction of matrices" Proc. Nat. Acad. Sci. USA 69(1972), 2530-2531, (with W. N. Anderson and G. E. Trapp).

A new Hermitian semidefinite matrix operation is studied. This operation - called parallel subtraction - is developed from the theory of paralled addition. Since the theory of parallel addition is motivated by the analysis of interconnected electrical networks, parallel subtraction may be interpreted in terms of the synthesis of electrical networks. The idea of subtraction is also extended to hybrid addition.

23. "Hilbert transforms in Yukawan potential theory"
Proc. Nat Acad. Sci USA 69(1972) 3677-3679.

If H denotes the classical Hilbert transform and $Hu(x) = v(x)$, then the functions $u(x)$ and $v(x)$ are the values on the real axis of a pair of conjugate functions, harmonic in the upper half-plane. This note gives a generalization of the above concepts in which the Laplace equation $\Delta u = 0$ is replaced by the Yukawa equation $\Delta u = \mu^2 u$ and in which the Cauchy-Riemann equations have a corresponding generalization. This leads to a generalized Hilbert transform H_μ. The kernel function of this new transform is expressable in terms of the Bessel function K_0. The transform is of convolution type.

24. "An integral equation formulation of Maxwell's equations"
Journal of Franklin Inst. 289(1974) 385-394 (with J. H. McWhirter)

A classical method for solving static field problems is based on Fredholm integral equations. Here we consider the applications of integral equations to the general electromagnetic problem when the applied fields are alternating. Attention is focused on a problem with cylindrical symmetry. By employing Green's third identity, the boundary value problem is turned into a pair of integral equations of the second kind. This set of equations can form the basis for the numerical solution of these problems.

25. "A differential equation describing biological cell growth inhibition"
(with J. Schubert).

A phenomenological model for the inhibition of cell growth by a chemical agent is formulated. This leads to a formula relating the cell doubling time to the concentration of the agent. This formula gives good agreement with experimental data. An extension of this formula is proposed to treat a mixture of agents.

26. "On Fourier's analysis of linear inequality systems"
Math. Programming Study 1(1974) 71-94.

Fourier treated a system of linear inequalities by a method of elimination of variables. This method can be used to derive the duality theory of linear programming. Perhaps this furnishes the quickest proof both for finite and infinite linear programs. For numerical evaluation of a linear program,

Fourier's procedure is very cumbersome because a variable is eliminated by adding each pair of inequalities having coefficients of opposite sign. This introduces many redundant inequalities. However, modifications are possible which reduce the number of redundant inequalities generated. With these modifications the method of Fourier becomes a practical computational algorithm for a class of parametric linear programs.

27. "Some problems of mathematics and science"
Bulletin of the American Mathematical Society 80(1974), 1053-1070.

The development of mathematics has often been aided by the use of models from science and technology. There are three main reasons why models help: (i) attention is focused on significant problems; (ii) the intuition is aided in perceiving complex relations; (iii) new concepts are suggested. This paper describes problems arising from models which have interested me. The models come from physics, chemistry, engineering, and economics.

28. "Nonlinear networks IIa"
Nonlinear Networks: Theory and Analysis, Edited by A. N. Wilson, IEEE Press (1975), 29-37.

This is a reprint of a paper on the steady flow of current in an electrical network. It is shown that if the resistors have a monotone characteristic then a unique flow exists.

29. "Lagrange multiplier method for convex programs"
Proc. Nat. Acad. Sci. USA 72(1975), pp. 1778-1781.

The problem of minimizing a convex function that is subject to the constraint that a number of other convex functions be non-positive can be treated by the Lagrange multiplier method. Such a treatment was revived by Kuhn and Tucker and further studied by many other scientists. These studies led to an associated maximizing problem on the Lagrange function. The aim of this note is to give a short elementary proof that the infimum of the first problem is equal to the supremum of the second problem. To carry this out it is necessary to relax the constraints of the first (or the second) problem so that the constraints are enforced only in the limit. This relaxation of constraints is not necessary should prescribing upper bounds to all the points in the domain of the functions. The domain of the functions can be n-dimensional space or a reflexive Banach space.

30. "Electrical network models"
Studies in Graph Theory, Edited by D. R. Fulkerson, MAA Studies 11(1975), pp. 94-138.

The history of science shows that the development of mathematics has been accelerated by the use of models. Thus geometric diagrams have served as models for algebraic relations. Gambling has served as a model for probability theory. Gravitation has served as a model for harmonic functions. Such models have accelerated mathematical development for three main reasons: (i) Attention is focused on significant problems. (ii) Models aid the intuition in perceiving complex relations. (iii) New concepts are suggested.

31. "Distortionless wave propagation in inhomogeneous media and transmission lines"
Quarterly of Appl. Math. 34(1976) pp. 183-194 (with V. Burke and D. Hazony).

Of concern are mechanical or electrical waves in a media which may be nonuniform and dissipative. The problem posed is to find conditions for the undistorted propagation of signals. The electrical transmission line is chosen as the general model. Along the length of the transmission line there are four functions which may be prescribed essentially arbitrarily. These are series resistance, series inductance, shunt conductance, and shunt capacitance. A differential equation is derived relating these functions which gives a necessary and sufficient requisite for distortionless transmission of a voltage wave. Various corollaries of this theorem are developed. For instance, it is shown that simultaneous voltage and current waves can be transmitted without distortion if and only if the characteristic impedance of the transmission line is positive at each point.

32. "Orbits of most action on a convex billiard table"
Houston J. of Math. 2(1976) pp. 453-470 (with L. A. Karlovitz).

Of concern are the trajectories of a ball bouncing inside a convex enclosure. The ball is treated as a mass point but the Coriolis force is not neglected. Thus between bounces a trajectory is a circular arc of given radius. At a bounce the angles of incidence and reflection are equal. Complete trajectories are analyzed from two different points of view. The first relies on a static model of the dynamic situation. The second is based on a version of the Jacobi least action principle and the notion of total action. By either analysis it is shown for example, that there are convex periodic orbits having a prescribed number of bounces. Another problem treated is the determination of a

Coriolis ellipse (a convex enclosure having two focal points).

33. "Convex programs having some linear constraints"
Proc. Natl. Acad. Sci. USA 74(1977) pp. 26-28.

The problem of concern is the minimization of a convex function over a normed space (such as a Hilbert space) subject to the constraints that a number of other convex functions are not positive. As is well known, there is a dual maximization problem involving Lagrange multipliers. Some of the constraint functions are linear, and so the Uzawa, Stoer, and Witzgall form of the Slater constraint qualifications is appropriate. A short elementary proof is given that the infimum of the first problem is equal to the supremum of the second problem.

34. "Algorithms for localizing roots of a polynomial and the Pisot Vijayaraghavan numbers"
Pacific J. of Math. 74(1978) pp. 47-56.

Pisot and Vijayaraghavan studied numbers whose mth power is nearly an integer in the sense that the discrepancy vanishes as m becomes infinite. One plus square root two is an example. Algebraic numbers of this type are characterized as algebraic integers whose conjugate roots each have absolute value less than one. This note develops a test for this property. An algorithm is given which determines whether or not one root of a polynomial has absolute value greater than one and all the other roots have absolute value less than one. If n is the degree of the polynomial, this algorithm involves only n rational steps.

35. "Almost definite operators and electro-mechanical systems"
SIAM J. Appl. Math. 35(1978) pp. 21-30 (with T. D. Morley).

In recent years the concept of parallel addition and other new operations have been introduced. These operations are derived from network interconnections involving Kirchoff's and Ohm's laws. From electrical considerations the domains of these operations, for the most part, have been restricted to positive semi-definite operators on a finite dimensional inner product space. In this paper we consider general electro-mechanical systems of the form: given a complex linear transformation $A : \mathcal{K} \to \mathfrak{U}$ and a complex linear operator Z on $\mathfrak{U}$, we seek a linear operator $\Psi(Z)$ on the range of A, such that for any $b \in$ range (A), there is a solution x, v to $Ax = B$, $Zx - A^*v = 0$ with $\Psi(Z) = v$. We term Z almost definite if $(Zx,x) = 0$ only when $Zx = 0$. If Z satisfies this condition, then $\Psi(Z)$, termed the transfer resistance, exists and is unique. Various properties are developed. In particular, an operator Z is almost definite if and only

if $\Psi(Z)$ exists for all possible A. If Z is almost definite, then there is a complex constant α with $|\alpha| = 1$, such that $\mathrm{Re}(\alpha Zx,x) \geq 0$, for all $x \in \mathfrak{U}$. This model extends the domain of validity of the new operations and unites the theory.

36. "Inequalities induced by network connections" Recent Applications of Generalized Inverses, Editor, N. Z. Nashed, Pitway, London (1980), (with T. D. Morley).

It has been found that interesting mathematical relationships arise from a vectorial generalization of Kirchhoff's and Ohm's laws, in which the "resistors" become positive semi-definite (PSD) linear operators. In analogy to the parallel connection of resistors Anderson and Duffin stidied the parallel sum $P : S$ of two PSD operators on a finite dimensional space, defined by $R : S = R(R+S)^{+}S$. This paper extends the results of Anderson and Duffin that $\|R:S\| \leq \|R\|:\|S\|$ and $\mathrm{tr}[R:S] \leq \mathrm{tr}R:\mathrm{tr}S$ to a wide class of operations derived by a vectorial analog of Kirchhoff's and Ohm's laws. These inequalities remain true in Hilbert space.

37. "Computational methods for solving static field and eddy current problems via Fredholm integral equations" IEEE Trans. on Magnetics 15(1979) pp. 1075-1084 (with J. H. McWhirter, J. J. Oravec and P. J. Brehm).

Two-dimensional static field problems can be solved by a method based on Fredholm integral equations (equations of the second kind). This has numerical advantages over the more commonly used integral equation of the first kind. The method is applicable to both magnetostatic and electrostatic problems formulated in terms of either vector or scalar potentials. It has been extended to the solution of eddy current problems with sinusoidal driving functions. The application of the classical Fredholm equation has been extended to problems containing boundary conditions: (1) potential value, (2) normal derivative value, and (3) an interface condition, all in the same problem. The solutions to the Fredholm equations are single or double (dipole) layers of sources on the problem boundaries and interfaces. This method has been developed into computer codes which use piecewise quadratic approximations to the solutions to the integral equatiosn. Exact integrations are used to replace the integral equations by a matrix equation, The solution to this matrix equation can then be used to directly calculate the field anywhere.

38. "Inequalities induced by network connections II. Hybrid connection J. of Math. Anal. and Appl. 67(1979) pp. 215 - (with T. D. Morley).

It has been found that interesting mathematical relationships arise from a vectorial generalization of Kirchhoff's and Ohm's laws, in which the "resistors" become Hermitian positive semidefinite (PSD) linear operators. In analogy to the parallel connection of resistors Anderson and Duffin studied the parallel sum $R : S$ of two PSD operators on a finite dimensional space, defined by $R : S = R(R+S)^{-1}S$. Duffin and Trapp then studied the hybrid connection. This paper generalizes some of their results to a much broader class of electrical connections.

39. "Wang algebra and matroids"
IEEE Trans. on Circuits and Systems, 9(1978) pp. 755-762 (with T. D. Morley).

Wang algebra is defined by three rules: i) $xy = yx$; ii) $x + x = 0$; and iii) $xx = 0$. K. T. Wang showed that these rules give a shortcut method for finding the joint resistance (or driving point resistance) of an electrical network. However, there are electrical systems more general than the Kirchhoff network. For these systems regular matroids replace networks. It is shown in this paper that Wang algebra is an excellent tool to develop properties of networks. Moreover the Wang shortcut method can still be used to find the joint resistance of an electrical network.

40. "An elementary treatment of Lagrange multpliers"
Extremal Methods and Systems Analysis, Ed. by A. V. Fiacco and K. O. Kortanek, Springer-Verlag, Berlin , 1980 pp. 357-373.

The problem of finding the infimum of a convex function $f(x)$ subject to the constraint that one or more convex functions $g(x)$ be non-positive can be treated by the Lagrange multiplier method. Such a treatment was revived by Kuhn and Tucker and further studied by many other scientists. These studies led to the following associated maximizing problem on the Lagrange function, $L = f(x) + \lambda\, g(x)$. First find the infinum of L with respect to x and then take the supremum with respect to λ, subject to $\lambda \geq 0$. The minimizing problem and the associated maximizing problem are termed dual programs. This paper is partly of an expository nature: The goal is to give a short and elementary proof that, under suitable qualifications, the infinum of the first program is equal to the supremum of the second program. The proof begins by using the Courant penalty function. No knowledge of linear programming is assumed. However the duality theorem for linear programs is a special case of the duality theorem for convex programs developed.

41. "Clark's theorem on linear programs holds for convex programs" Proc. Natl. Acad. Sci. USA 75(1978) pp. 1624-1626.

Given a linear minimization program, then there is an associated linear maximization program termed the dual. F. E. Clark proved the following theorem. "If the set of feasible points of one program if bounded, then the set of feasible points of the other program is unbounded." A convex program is the minimization of a convex function subject to the constraint that a number of other convex functions be nonpositive. As is well known, a dual maximization problem can be defined in terms of the Lagrange function. The dual objective function is the infim of the Lagrange function. The feasible Lagrange multipliers are those satisfying: (i) the multipliers are nonnegative and (ii) the dual objective function is not negative infinity. It is found that Clark's Theorem applies unchanged to dual convex programs. Moreover, the programs have equal values.

42. "A class of optimal design problems with linear inequality constraints" Instituto Lombardo (Rend. Sc.) A 112 (1980) (with B. D. Coleman, and G. P. Knowles).

The optimal design of a structure often requires the minimization of a linear functional subject to integral inequali constraints. In this note, cases are treated in which the constraints are linear. One such problem is the long vertical cables of variable cross-section. Another is the cantilevered beams of variable width. The relation to linear programming theory is developed.

43. "Operator networks treated by Sylvester unisignants" (with T. D. Morley)

Sylvester proposed a special format for writing a system of n linear equations on n variables. It resulted that the matrix elements of the inverse are unisignants. A unisigna is a rational function of the n^2 coefficients formed by additi multiplication, and division, Subtraction is not permitted. We extend his theory to the case in which the coefficients are linear operators rather than scalars. The equations defining the steady flow of current through a network of conductors is a Sylvester system. By use of unisignants we generalize several properties of scalar networks to the operator case.

44. "Linear Inequality Systems Treated by Determinants" (with J. T. Buckwalter).

This study concerns a linear inequality system defined in terms of a matrix A. Fourier proposed to solve such a system by eliminating variables one at a time. Here Fourier's method is generalized so as to be able to eliminate variables in blocks rather than one by one. This solution is expressed in terms of certain minors of A. The dual problem of a system of equations with matrix A^T and positive variables is also studied. The solution is expressed in terms of the same minors of A by an algorithm analogous to Cramer's rule.

45. "Lagrangean Functions and Affine Minorants" (with R. G. Jeroslow).

We give hypotheses, valid in reflexive Banach spaces (such as L^p for $\infty > p > 1$ or Hilbert spaces), for a certain modification of the ordinary lagrangean to close the duality gap, in convex programs with (possibly) infinitely many constraint functions.

Our modification of the ordinary lagrangean is to perturb the criterion function by a linear term, and to take the limit of this perturbed lagrangean, as the norm of this term goes to zero.

We also review the recent literature on this topic of the "limiting lagrangean."

46. "Some problems arising from mathematical models" Constructive Approaches to Mathematical Models, Ed. by C. V. Coffman and G. J. Fix, Academic Press Inc. 1979 pp. 3-32.

In addition to previous models this paper describes the following problems:

(1) Transmission of signals without distortion.
(2) Wave motion and the Paley-Wiener theorem.
(3) Sylvester unisignant operators.
(4) A determinant algorithm for linear inequalities.
(5) Clark's theorem extended to convex programs.

47. "The fundamental mode of vibration of a clamped annular plate is not of one sign" Constructive Approaches to Mathematical Models, 1979 pp. 267-277. (with C. V. Coffman, and D. H. Shaffer).

We are concerned here with the modes of vibration of a clamped annular plate; that is to say, with the eigenvalue problem

$$\Delta^2\varphi = \lambda\varphi \quad \text{in} \quad D_\epsilon, \quad \varphi = \frac{\partial\varphi}{\partial n} = 0 \quad \text{on} \quad \partial D_\epsilon , \qquad (1.1)$$

where

$$D_\epsilon = \{z = x + iy : \epsilon < |z| < 1\}, \quad \epsilon > 0 .$$

and $\partial/\partial n$ denotes differentiation with respect to the outer normal.

Apparently is was A. Weinstein who first raised the question of whether the fundamental mode of vibration of a clamped plate must in general be of one sign. Interest in this question increased when Szegö showed that an affirmative answer would imply that among all plates of a given area the fundamental eigenvalue is smallest for a circular plate.

48. "Theory of monotonic transformations applied to optimal design problems"
Archive for Rational Mech. and Anal. 72(1980), 381-393 (with B. D. Coleman and G. Knowles).

The optimal design of a structure often requires the minimization of the value of an objective functional f over a set G of non-negative functions ζ which obey a constraint of the form $T\zeta(y) \leq \zeta(y)$, with T an integral operator. In a recent article we discussed cases in which the functional f and the operator T are affine functions. Here we present a method which can be useful when f and T are not affine but are monotonic in the sense that, for each pair of functions ζ_1 and ζ_2 in G obeying $\zeta_2(y) \geq \zeta_1(y)$ for almost all y, there holds $f(\zeta_2) \geq f(\zeta_1)$ and $T\zeta_2(y) \geq T\zeta_1(y)$. The method rests on a theorem of the following type: Under the assumptions that T is continuous in an appropriate sense and that the feasible set G, although not empty, does not contain the zero function 0, the infimum of f over G is attained at a function ψ, which is simultaneously a minimizer of f and a fixed point of T, can be obtained by successive applications of T to the zero function, i.e., $\psi = \lim T^n 0$, for almost all $y, T^n 0(y) \uparrow \psi(y)$. If the fixed point of T is unique, then, for each ζ in G, $\psi = \lim T^n\zeta$, and $T^n\zeta(y) \downarrow \psi(y)$ for almost all y.

49. "The limiting Lagrangean"
(with R. G. Jeroslow).

A somewhat modified form of the lagrangean closes the

duality gap in convex optimization, in many circumstances when the ordinary lagrangean and the augmented lagrangeans leave a duality gap. For example, the duality gap for a consistent program is always zero using this modified lagrangean, when the objective function and constraints are closed, convex functions; in other instances, there are constraint qualifications, but these are typically weaker than the usual Slater point requirements.

50. "Are adobe walls optimal phase shift filters?" Advances in Applied Math (1980) (with C. V. Coffman and G. P. Knowles), Vol. 1, pp. 50-66.

The adobe house construction gives an automatic, air conditioning effect because the rooms tend to be cool at midday and warm at night. Presumably this is brought about by the walls acting as a heat filter so that there is nearly a twelve hour phase lag. This raises the question of how to optimize the adobe phenomena by a suitable design of the walls.

In this study it is supposed possible to make the walls of layered construction with layers having different thermal resistivity. Such a layered wall can be modeled electrically as a ladder filter of capacitors and resistors. The input to the ladder is a sinusoidal voltage. Then the following question arises. If the filter capacitors have given values how should the resistors be chosen so that the output voltage has a given phase lag but least attenuation?

It is found possible to answer this question by use of a special variational principle. Applying this analysis to building construction shows how to maximize oscillation of interior temperature with a phase lag of a prescribed number of hours.

51. "A limiting Infisup theorem" Jour. of Optimization Theory 37 (1982) pp. 163-175 (with C. E. Glair and R. G. Jeroslow)

We show that duality gaps can be closed under broad hypotheses in minimax problems, provided certain changes are made in the maximin part which increase its value. The primary device is to add a linear perturbation to the saddle function, and send it to zero in the limit. Suprema replace maxima, and infima replace minima. In addition to the usual convexity-concavity assumptions on the saddle function and the sets, a form of semi-reflexivity is required for one of the two spaces of the saddle functions.

A sharpening of our result is possible when one of the spaces is finite-dimensional.

A variant of the proof of the previous results leads to a generalization of a result of Sion, from which the theorem of Kneser and Fan follows.

52. "Bounds for the r^{th} characteristic frequency of a beaded string or of an electrical filter"
Proc. Nat. Acad. Sci. (1980) (with M. F. Barnsley), pp. 3120-3124.

The fundamental mode of vibration of a beaded string has a shape without change of sign. The r^{th} higher normal mode of vibration has r changes of sign. Given any virtual shape of the string with r changes of sign, an algorithm is found which gives upper and lower bounds for the r^{th} characteristic frequency as a function of the virtual shape. By making a transformation it is found that this algorithm holds for the characteristic frequencies of an L-C network. Other transformations show that it applies to the r^{th} eigenvalue of a Hermitian matrix.

53. "Temperature control of buildings by wall design"
(with G. Knowles), Solar Energy 27 (1981), pp. 241-249.

The adobe house developed in the hot arid climate of the American Southwest has the virtue of being cool in the day and warm in the night. The adobe wall acts as a filter giving nearly a twelve-hour phase lag in the outside temperature oscillation. However, on reaching the inside, the oscillation suffers a strong attenuation in amplitude. In a previous paper it was shown that if the resistivity of the wall could be varied in a certain way from outside to inside then the attenuation would be considerably reduced. In this paper a formula is given for the minimum possible attenuation under given design restrictions. Then it is shown how to approximate the optimum wall by constructing a wall with two or more layers of different materials. Numerical illustrations are given.

54. "On the structure of biharmonic functions satisfying the clamped plate conditions on a right angle"
(with C. V. Coffman), Adv. Appl. Math. 1 (1980) pp. 373-389.

Let $u(r,\theta)$ be biharmonic and bounded in the circular sector $|\theta| < \frac{\pi}{4}$, $0 < r < \rho$ $(\rho > 1)$ and vanish together with $\frac{\partial u}{\partial \theta}$ when $|\theta| = \frac{\pi}{4}$. We consider the transform $\hat{u}(p,\theta) = \int_0^1 r^{p-1} u(r,\theta)\,dr$. We show that for any fixed θ_o $u(p,\theta_o)$ is meromorphic with no real poles and cannot be entire unless $u(r,\theta_o) \equiv 0$. It follows then from a theorem of Doetsch that $u(r,\theta_o)$ either vanishes identically or oscillates as $r \to 0$.

55. "Puzzles, Games and Paradoxes"
Mathematics Department Booklet, Carnegie-Mellon University, January, 1979.

Questions for a problem seminar in applied mathematics. Most of the questions are variants of old problems but some are new.

56. "On the relationshop between the genus and the cardinality of the maximum matchings of a graph"
Discrete Mathematics 25(1979) pp. 149-156 (By T. Nishizeki).

Lower bounds on the cardinality of the maximum matchings of graphs are established in terms of a linear polynomial of $p, p^{(1)}, p^{(2)}$ and γ whose coefficients are functions of k, where p is the number of the vertices of a graph, $p^{(i)}$ the number of the vertices of degree $i (i = 1,2)$, γ the genus and k the connectivity.

57. "Lower bounds on the cardinality of the maximum matchings of planar graphs"
Discrete Mathematics 28(1979) (By T. Nishizeki and I. Baybars).

Lower bounds on the cardinality of the maximum matchings of planar graphs, with a constraint on the minimum degree, are established in terms of a linear polynomial of the number of vertices. The bounds depend upon the minimum degree and the connectivity of graphs. Some examples are given which show that all the lower bounds are best possible in the sense that neither the coefficients nor the constant terms can be improved.

58. "Linear Programming Supply Models for Multihour Traffic Networks"
Modeling and Simulation, edited by M. H. Mickle and W. G. Vogt. Socio-Eco. Systems, Instrument Society of America 10, part 4, 1431-1435, (By P. R. Gribik, K. O. Kortanek, D. N. Lee and G. C. Polak).

Marginal Investment costs are assumed known for circuits and switching facilities employed to supply telecommunications services. Poisson generated customer demands are specified between junction pairs in a network hierarchy by times of day. A linear programming supply model is presented for computing circuit group sizes.

59. "Hewlitt-Nachbin spaces"
North-Holland/American Elsevier, Amsterdam 1975 (By Maurice Weir)

The class of Hewitt-Nachbin spaces has won the interest of many point-set topologists in the past quarter-century. This book summarizes the recent progress. It serves to classify and to solve problems in a variety of mathematical discplines.

60. "Mathematical Methods in the Social and Managerial Sciences"
John Wiley and Sons, Inc. 1975 (By Patrick Hayes).

An introduction to differential equations, convexity, convex programming and geometric programming. Many of the problems are posed in terms of business or economic concepts.

61. "Tranformation that aid numerical solution of nonlinear programs,"
Opsearch 18 (1981) pp. 158-161.

The main goal of this note is to transform programs so that it is easier to find feasible solutions. Inverse programming is studied.

62. "Numerical eseimation of optima by use of duality inequalities,"
Proc. International Symposium on Semi-Infinite Programming (1982).

In order to use a duality inequality to estimate the minimum value of a convex program, no equality constraints should appear in the program or the dual program. The reason for this is that it is impractical to find feasible solutions to non-linear equations. Unfortunately the usual Lagrange type dual program has equality constraints. To avoid this difficulty a modified dual program is developed by using suitable support planes. Only inequality constraints then arise.

63. "Uniform duality in semi-infinite linear programming,"
(with R. G. Jeroslow and L. A. Karlovitz), Proc. International Symposium on Semi-Infinite Programming (1982).

A duality theorem is developed which holds for any objective function. Moreover the duality theorem, so stated, reduces to the classical theorem if this program is finite.

65. "Rubel's universal differential equation," Proc. Nat. Acad. Sci. 78 (1981) pp. 4661-4662.

The concept of a universal partial differential equation was proposed by Creighton Buck of the University of Wisconsin. Lee Rubel of the University of Illinois considered the case of ordinary differential equations and found a universal differential equation of polynomial form having seven terms. Later we found the following example having only three terms.

$$(*) \qquad 2y''''y'^2 - 5y'''y''y' + 3y''^3 = 0$$

Given a continuous function φ on the real axis and $\epsilon > 0$. Then (*) has a solution which approximates φ within an error less than ϵ over the whole axis.

XIII. Research in Progress (1983)

This is the final report of the research project "Analysis of Electrical and Mechanical Systems." However scientific research often has a phoenix property - ours is no exception. We list here six promising new topics, A, B, C, D, E, F:

A. Telephone network optimization

After several years time the volume of calls may increase to the extent of causing unacceptable call delays. In that case the network must be redesigned with increased channel capacitites. This entails a modified switching system as well. The cost of the redesign is essentially the cost of the channels plus that of the switching. Dr. Kenneth Kortanek and Dr. George Polak have been studying this design problem. (Dr. Polak was a research assistant with the AEMS Project.)

The goal is to integrate queueing theory of user behavior with a multicommodity network model for a telecommunication system. The approach to be developed is to compute a probability distribution on the number of trunk-lines demanded. This approach is more precise than the use of "blocking functions".

The economical design will then be posed as both (1) a chance constrained program and (2) a stochastic program with recourse. Then a solution of the latter will be attempted using the so called cost operator algorithm.

B. The infiltration problem in passive solar heating

The daily variation in outdoor temperature causes a related variation at the inside surface of the walls of an unheated building. However, the speed of the heat wave flowing through the wall is very slow so if the wall is sufficiently thick there will be a delay of around 12 hours in the transmission of the high and low temperature. This time lag of peak temperature can be of practical value in hot climates. In the last two years we published two papers showing how to design a layered wall so as to optimize the cooling effect.

We now propose to consider more moderate climates where the average daily temperature may be a comfortable living condition. If the house were perfectly sealed this would simply require sufficient insulation in the walls. In practice, however, there will be air leaks through doors and windows. Since infiltration losses are essentially in phase with the outside temperature fluctuation, we can use the lag in the temperature flow through the walls to offset them.

The problem to be studied is the design of a layered wall so as to minimize the fluctuation of temperatures in the interior of the building. The mathematical questions appear somewhat similar to those in our previous papers. We hope to devise a similar algorithm to give an optimum design.

C. Phase shift networks

To analyze the flow of heat in our first paper on the adobe wall we used a ladder RC filter as a model of the wall. A solution of the filter was sought by varying the resistors to mimimize the attentuation of a sinusoidal wave subject to a given phase lag at the output.

It is proposed to generalize these previous developments in two ways. First a general RC network is to be the model. However all capacitors are to have one grounded terminal. This property is required if voltage is to be analogous to temperature. The second generalization is to vary the capacitors rather than the resistors. Preliminary analysis suggests that similar theorems and algorithms hold.

D. Calculus of variations with complex functions

Our study of heat flow through layered walls was carried out mainly by modeling the wall as a ladder filter of n meshes. Of course such a filter is a discrete system. A difficulty in dealing directly with the differential equation of heat flow is that the temperature function $u(x)$ is complex valued. There is very little literature on calculus of variations with complex functions.

We found that an optimal solution u for the ladder filter satisfies the following condition, uu^* is real at each node of the filter. Here u^* is also a solution of the filter but with different boundary conditions. We termed the above statement "the equiphase theorem".

The proposed study would try to find a direct rigorous proof that $u(x)u^*(x)$ was real valued for $0 \leq x \leq 1$. No doubt other interesting questions would then arise.

E. Eddy current theory and application

Consider an apparatus S which is a source of a periodic magnetic field. When a conducting body B is placed near S then periodic electric currents are induced in B. They are called eddy currents. This is a typical situation which arises in science and engineering. In some cases the eddy currents are desirable as in the induction furnace. In other cases they are undesirable as they waste energy or cause overheating.

The determination of the eddy currents can be achieved by applying Maxwell's equations. The problem is then reduced to a boundary value problem for a vector potential A which satisfies the equation

$$(*) \qquad \Delta A = kA$$

Here Δ is the Laplacian operator and k is a function of position depending on the conductivity of body B. If the source S has "low frequency" then k is pure imaginary.

If k were zero then (*) is the Laplace equation and if k were negative (*) is the Helmholtz equation. In both of these cases the literature is very extensive. However for k positive or complex not so much has been written on the properties of equation (*).

If k is positive (*) is the Yukawa equation and two previous studies of the AEMS Project concern this equation:

"Yukawan potential theory", J. Math. Anal. and Appl. 35 (1971), pp. 105-131,

"Hilbert transforms in Yukawan potential theory", Proc. Nat. Acad. Sci. 64 (1972), pp. 3677-3679.

In these papers attention was confined to the two-dimensional case and an analogy was developed with the well known case, $k = 0$. In particular the analogs of conjugate harmonic functions were studied by means of Cauchy-Riemann type equations.

The proposed research concerns k pure imaginary. Research in this area was initiated by Lord Kelvin who introduced the Ber and Bei type Bessel functions to treat the skin effect when alternating current flows along a cylindrical conductor. Since the time of Kelvin many practical problems relating to eddy currents have been solved by engineers. However there has been little effort to develop a systematic study of equation (*) with k imaginary.

The proposed research with k imaginary is to begin by developing analogy and counter analogy with the classical theory $(k = 0)$. This will include the Dirichlet principle, the Gauss mean value theorem, the maximum principle, conjugate functions, and Hilbert transforms.

Some preliminary studies have already been made. It appears that satisfactory analogs for some of the above theorems do exist. Moreover it seems that this theory will supply answers to practical questions about eddy currents.

F. The finite surface element method

Fredholm used his integral equation to prove that the classical boundary value problem of potential theory has a solution. The same equation can be used to obtain a numerical solution of the problem. For example consider a polyhedral region. Decompose the surface of the region into triangles. Given a continuous function on the surface and suppose it has quadratic variation in each triangle. Such functions are used to approximate the dipole density on the surface of the region. To employ the Fredholm method effectively it is necessary to have an analytical formula giving the newtonian potential of a triangle with such a charge density. The integrals involved are complicated but can be evaluated in terms of elementary functions. We propose to express such integrals in a form convenient for numerical use.

The Fredholm integral equation method can also be applied to the two dimensional case of Maxwell's equations. The theory for this has already been developed in a previous ARMS study:

> "An integral equation formulation of Maxwell's equations", Jour. Franklin Inst. 298 (1974), 385-394.

However this paper did not show how to resolve the integral equation on a computer. Thus the kernel of the integral equation is a Bessel function. Consequently algorithms must be be developed to evaluate this kernel. Also a finite element scheme must be designed to evaluate the boundary integrals. It is hoped to have a research assistant carry out these questions of numerical analysis.

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